MATHEMATICS 2º ESO

SECCIONES EUROPEAS

IES ANDRÉS DE VANDELVIRA

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INDEX

1 Numbers, operations, integers, divisibility

1 Remember how to read numbers

Complete this table

Number	Cardinal	Ordinal
1	One	first (1^{st})
2	Two	second (2^{nd})
3	Three	third (3^{rd})
4		fourth (4^{th})
5		fifth
6		sixth
7		seventh
8		eighth
9		ninth
10		tenth
11	Eleven	eleventh
12	Twelve	
13	Thirteen	
14	Fourteen	
15	Fifteen	
16	Sixteen	
17	Seventeen	
18	Eighteen	
19	Nineteen	
20	Twenty	
21	twenty-one	twenty-first
22		
23		
24		
25		
26		
27		
28		
29		
30	Thirty	thirtieth
40	Forty	
50		
60		
70		

80		
90		
100	one hundred	hundredth
1,000	one thousand	
100,000		hundred thousandth
1,000,000	one million	millionth

The names of the big numbers differ depending where you live. The places are grouped by thousands in America, or France, by millions in Great Britain (not always), Germany and Spain.

Name	American-French	English-German-Spanish
million	1,000,000	1,000,000
billion	1,000,000,000 (a thousand millions)	1,000,000,000,000 (a million millions)
trillion	1 with 12 zeros	1 with 18 zeros

We will read the numbers in our own way although we must be capable of recognize the correct meaning of numbers when the information comes to us from USA, for example.

Decimals

Decimal fractions are said with each figure separate. We use a full stop (called **“point”**), not a comma, before the fraction. Each place value has a value that is one tenth of the value to the immediate left of it.

0.75 (nought) **point** seventy-five or zero **point** seven five

3.375 three **point** three seven five.

We will see all this more detailed in chapter 2

2 Fractions and percentages

Simple fractions are expressed by using *“ordinal numbers”* / (third, fourth, fifth..) with some exceptions:

1/2 One half / a half
1/3 One third / a third
2/3 Two thirds
3/4 Three quarters
5/8 Five eighths
4/33 Four **over** thirty-three

Percentages:

We don’t use the article in front of the numeral

10% of the people Ten **per cent** of the people

We will see this more detailed in chapters 3 and 5

Exercise 1

a) Write in words the following numbers as in the example.

3 528: Three thousand, five hundred and twenty eight

86 424 ______________________________

987 ______________________________

3 270 ______________________________

30 001 ______________________________

1 487 070 ______________________________

320 569 ______________________________

20,890,300 ______________________________

b) Read aloud the following numbers:

456	**4 500**	**90 045 123**
34 760 041	**23 455 678**	**5 223 500 668 316**

c) Write in words the following decimal and fractions:

0.056 ______________________________

23.45 cm ______________________________

£1.20 ______________________________

3.77 ______________________________

$\frac{5}{4}$ ______________________________

$\frac{7}{2}$ ______________________________

$\frac{12}{35}$ ______________________________

$\frac{37}{100}$

3 Reading powers

6^5 Is read as

- **The fifth power of six**
- **Six to the power of five**
- **Six powered to five**.

The most common is **six to the power of five**

6 is the **base**

5 is the **index** or **exponent**

Especial cases:

Squares and cubes (powers of two and three)

3^2 is read as:

- **Three squared**
- **Three to the power of two.**

5^3 Is read as:

- Five cubed
- Five to the power of three.

The most common is **five cubed**

Exercise 2 Calculate mentally and write in words the following powers:

a) $4^3 =$ ______________________

b) $6^2 =$ ______________________

c) $(-11)^2 =$ ______________________

d) $2^5 =$ ______________________

e) $10^3 =$ ______________________

f) $1000^2 =$ ______________________________

g) $(0.1)^3 =$ ______________________________

4 Calculations

Addition AND / PLUS

In small additions we say ***and*** for addition and ***is/are*** for the result

Example:

2+6 = 8 Is read as "two **and** six **are** eight" or "two **and** six **is** eight"

In larger additions and in **more formal style** (in maths) we use ***plus*** for **+**, and ***equals*** or ***is*** for the result.

Example:

We read 234 + 25 = 259 like "two hundred and thirty four plus twenty five **equals** / **is** two hundred and fifty nine

Subtraction: MINUS/TAKE AWAY/FROM

Example:

$\mathbf{9}-\mathbf{5}=\mathbf{4}$

In conversational style, with small numbers, people say:

- Five **from** nine **leaves/is** four

- Nine **take away** five **leaves/is** four

In a more formal style, or with larger numbers, we use ***minus*** and ***equals***

510 - 302 = 208 Five hundred and ten **minus** three hundred and two **equals** /**is** two hundred and eight

Multiplication **TIMES** **MULTIPLIED BY**

In small calculations we say:

3 x 4 = 12 three fours **are** twelve
6 x 7 = 42 six sevens **are** forty-two

In larger calculations we can say
17 x 381 = 6,477

17 **times** 381 **is/makes** 6,477

In a more formal style:

17 **multiplied by** 381 **equal** 6,477

Division

DIVIDED BY

270:3 = 90   Two hundred and seventy **divided by** three **equals** ninety

But in smaller calculations like 8:2 = 4 we can say two **into** eight **goes** four (times)

5 Negative numbers

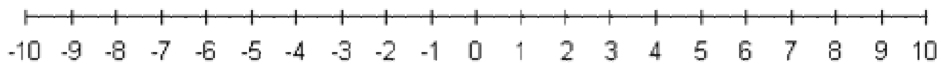
There are many situations in which you need to use numbers below zero, one of these is temperature, others are money that you can deposit (positive) or withdraw (negative) in a bank, steps that you can take forwards (positive) or backwards (negative).

Positive integers are all the whole numbers greater than zero: 1, 2, 3, 4, 5, ...

Negative integers are all the opposites of these whole numbers: -1, -2, -3, -4, -5, …

The Number Line

The number line is a line labelled with the integers in increasing order from left to right, that extends in both directions:



For any two different places on the number line, the integer on the right is greater than the integer on the left.

Examples:

9 > 4 Is read: nine is **'greater than'** four -7 < 9 Is read: minus seven is **'less than'** nine.

5.1 Absolute Value of an integer

The number of units a number is from zero on the number line.

If the number is positive, the absolute value is the same number.

If the number is negative, the absolute value is the opposite.

The absolute value of a number is always a positive number (or zero). We specify the absolute value of a number *n* by writing *n* in between two vertical bars: $|n|$.

Exercise 3 Plot on the number line and after order them from less to great.

- 2 + 8 0 - 5 3

5.2 Adding Integers

Rules for addition:

When adding integers **with the same sign:**

We add their absolute values, and give the result the same sign.

With the opposite signs:

We take their absolute values, subtract the smaller from the larger, and give the result the sign of the integer with the **larger** absolute value.

5.3 Subtracting Integers

Rules for subtraction:

Subtracting an integer is the same as adding the opposite.

We convert the subtracted integer to its opposite, and add the two integers.

The result of subtracting two integers could be positive or negative.

Exercise 4 Calculate operating first the expressions into brackets.

a) $8-(3-2+5)$ **b)** $87-4-9)+(2+3-1)$

c) $(2-7)-(5-4)-(1-6)$ **d)** $(2+4-1)-(7-4-9)$

e) $-(1-2+3-7)+(3-4)-(2-3-5)$

Exercise 5 Remove brackets and calculate:

a) $8-(3-2+5)$ **b)** $87-4-9)+(2+3-1)$

c) $(2-7)-(5-4)-(1-6)$ **d)** $(2+4-1)-(7-4-9)$

e) $-(1-2+3-7)+(3-4)-(2-3-5)$

Exercise 6 Calculate

a) $5-4+7-9+12-1+5-10$

b) $4-3+21-32-5+6$

c) $5-[2-(6-9)]$

d) $5-[5+(3-2-9)]-(3-10)$

e) $1-[(2-5)-3-(9-1)]-(2-3)$

5.4 Multiplying Integers

To multiply a pair of integers:

If both numbers have **the same sign** (positive or negative)

Their product is the product of their absolute values (their **product is positive**)

If the numbers have **opposite signs**

Their product is the *opposite* of the product of their absolute values (their product **is negative**).

If one or both of the integers is 0, the product is 0.

Look at the following chart below.

PRODUCT	+	-
+	POSITIVE	NEGATIVE
-	NEGATIVE	POSITIVE

To multiply any number of integers:

1. Count the number of negative numbers in the product.

2. Take the product of their absolute values.

If the number of negative integers counted in step 1 is **even**, the product is just the product from step 2 (**positive).**

If the number of negative integers is **odd,** the product is the opposite of the product in step 2 (give the product in step 2 a **negative** sign).

If any of the integers in the product is 0, the product is 0.

5.5 Dividing Integers

To divide a pair of integers the rules are the same than for the product:

If both numbers have the same sign (positive or negative)

Divide the absolute values of the first integer by the absolute value of the second integer (the result is positive)

If the numbers have opposite signs

Divide the absolute value of the first integer by the absolute value of the second integer, and give this result a negative sign.

The chart is.

DIVISION	+	-
+	POSITIVE	NEGATIVE
-	NEGATIVE	POSITIVE

6 Order of operations

Do all operations in brackets first.

Then do multiplications and divisions in the order they appear.

Then do additions and subtractions in the order they occur

Easy way to remember them

Brackets	Exponents	Divisions	Multiplications	Additions	Subtractions

This gives you: **BEDMAS.**

Do one operation at a time.

Exercise 7 Calculate

a) $(-3)(-2)(-5)$

b) $6\cdot(-3)\cdot(-1)$

c) $15:(-3)\cdot(-1)$

d) $(-80):[(-8)\cdot 2]$

e) $[(-80):(-8)]\cdot 2$

f) $[9\cdot(-8)]:[(-3)\cdot(-4)]$

Exercise 8 Calculate

a) $3(5-8)+2(8-12)-3+3\cdot(-5)$

b) $17-[2\cdot 3+5(-2+4)]\cdot 3-6\cdot(-5)$

c) $(-2)^2-(-3)^2+(-1)^3$

d) $(+4)^2+(-4)^2-(-4)^2-4^2$

e) $2^3-3^2+4^2-1^7$

f) $[5^2\cdot(-3)^2]:(-15)$

Exercise 9 Calculate operating first the expressions into brackets

a) $3(2-6+5)+7(3-2(5-12))$

b) $2\cdot[-3\cdot(5\cdot(3-4)+1)-2]$

c) $(-2)^3\cdot[-2+3\cdot(1-7)]:(1-3)$

d) $3\cdot(5-8)+2\cdot(3-12)-5+3\cdot 2$

---

7 Multiples and factors

7.1 Multiples

The products of a number with the **natural numbers** 1, 2, 3, 4, 5, ... are called the **multiples** of the number.

The multiples of a number are obtained by multiplying the number by each of the natural numbers.

7.2 Factors

A whole number that divides exactly into another whole number is called a **factor** of that number.

If a number can be expressed as a product of two whole numbers, then the whole numbers are called **factors** of that number.

7.3 Prime Numbers

If a number has only two different factors, 1 and itself, then the number is said to be a **prime number.**

The Sieve of Eratosthenes

Have a look of the book of 1º ESO and you will see how to get a set of all the prime numbers
Exercise 10 Write down all the prime numbers between 80 and 110 (you must use the Sieve of Eratosthenes).

7.4 Tests of divisibility

One number is divisible by:

2 If the last digit is 0 or is divisible by 2, (0, 2, 4, 8).

3 If the sum of the digits is divisible by 3.

4 If the last two digits are divisible by 4.

5 If the last digit is 0 or is divisible by 5, (0,5).

9 If the sum of the digits is divisible by 9.

8 If the half of it is divisible by 4.

6 If it is divisible by 2 and 3.

11 If the sum of the digits in the even position minus the sum of the digits in the uneven position is 0 or divisible by 11.

Exercise 10 Write down four consecutive multiples of:

a) 7 greater than 100

b) 15 greater than 230

c) 9 greater than 1230

Exercise 11 Write down all the multiples of 6 between 92 and 109

Exercise 12 Write down all the multiples of 6 between 1200 and 1250

Exercise 13 Write down all the factors of

a) 18

b) 90

c) 140

d) 80

e) 50

Exercise 14 Find out the missing figure so the number (there can be more than one answer)

a) 3[ ]1 is a multiple of 3

b) 57[ ] is a multiple of 2

c) 23[ ] is a multiple of 5

d) 52[ ]3 is a multiple of 11

Exercise 15 Factorise:

a) 46 b) 180 c) 60 d) 1500 e) 135

7.5 Common Multiples

Multiples that are common to two or more numbers are said to be **common multiples**.

Example:

Multiples of 2 are 2, 4, **6**, 8, 10, **12**, 14, 16, **18**, …

Multiples of 3 are 3, **6**, 9, **12**, 15, **18**, …

So, common multiples of 2 and 3 are 6, 12, 18, …

Lowest common multiple

The smallest common multiple of two or more numbers is called the **lowest common multiple** (LCM).

Example:

Multiples of 8 are 8, 16, **24**, 32, …

Multiples of 3 are 3, 6, 9, 12, 15, 18, 21, **24**, …

LCM of 3 and 8 is 24

In general there are two methods for finding the lowest common multiple (LCM) of two or more numbers**:**

Method I

For small numbers

List the multiples of the largest number and stop when you find a multiple of the other number. This is the LCM.

Method II

General

To find the lowest common multiple (LCM) of two or more higher numbers:

- Find the prime factor decomposition.

- Choose the **non common factors** and the **common factors with the highest exponents.**

Example: Find the lowest common multiple of 18 and 24.

$18 = 2 \cdot 3^2$
$24 = 2^3 \cdot 3$
So, the LCM of 18 and 24 is $LCM = 2^3 \cdot 3^2 = 72$.

7.6 Common Factors

Factors that are common to two or more numbers are said to be **common factors**.

Highest Common Factor

The largest common factor of two or more numbers is called the **highest common factor** (**HCF**).

Method I

For small numbers

For example:
$8 = 1x8 = 2x4$
$12 = 1x12 = 2x6 = 3x4$

- Factors of 8 are 1,2, 4 and 8
- Factors of 12 are 1, 2, 3, 4, 6 and 12

So, the common factors of 8 and 12 are 1, 2 and 4 HCF is **4**

Method II

General

To find the **Highest Common Factor** of two or more higher numbers:

- Find the prime factor decomposition.

- Choose only the common factors with the lowest exponents.

Exercise 16 Factorise and then calculate the LCM and the HCF of these numbers:

a) 360 and 300

b) 168 and 490

c) 12, 100 and 6

d) 14112, 1080 and 1008

e) 1600 and 1200

f) 294, 1050 and 28

Exercise 17 We want to distribute 100 l of water in bottles which all have the same capacity. Find out all the different solutions. Indicate how many bottles we get in each case and the capacity of each one

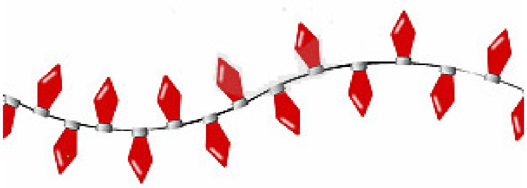
Exercise 18 Sandra can pack her books in boxes of 5, 6 and 9. She has less than 100 books. How many books has she got?

Exercise 19 We want to divide a rectangle of 600cm by 90 cm into equal squares. Find out the length of the biggest square in cm. Calculate how many squares we get.

Exercise 20 Iberia has a flight from Madrid to Ankara every 8 days, British Airways one every 12 days and Easy Jet one every 6 days; one day all three have a flight to Ankara. After how many days will the three flights coincide again?

Exercise 21 A group of students can be organized in lines of 5, 4 and 3 students and there are less than 100 students. How many are there?

Exercise 22 On a Christmas tree, there are two strings of lights, red lights flash every 24 seconds and green lights every 36 seconds. They start flashing simultaneously when we connect the tree. When will they flash together again?



2 Decimal and sexagesimal system

Keywords

Regular numbers **repeating decimals** **period**
irrational number **rounding** **number line**
sexagesimal system

1 Decimal numbers

Remember that to express numbers that are not whole numbers we use decimal numbers as 75.324 in which every digit has a value which is divided by ten when we move to the right. So

7 is seventy units

5 is five units

3 is three tenths of a unit

2 two hundredths of the unit

4 is four thousandths of a unit

And we continue like that if there are more digits.

This is the decimal system that is commonly use nowadays except, sometimes, for time and angles.

We read these numbers naming the whole part then "point" and then the decimals digits one by one

Example The number 75.324 is read as seventy five, point, three, two, four

When the numbers express money or length can be read on a different way, for example 5.24€ is read as five point twenty four euros or five euros and twenty five cents or the number 5.36 m can be read as five point thirty six metres.

2 Types of decimal numbers

As a result of some operations we can get different types of decimal numbers:

Regular numbers: Are decimal numbers with a limited quantity of decimal digits and from them all could be zeros.

Example $\frac{14}{5} = 2.8$ we find an exact division

Repeating decimals: There is a group of digits that are repeated forever.

Examples

If we divide 4 by 3 we get 1.3333333…

Calculating $\frac{13}{33} = 0.36363636...$

The group of repeated decimal digits is called the **period** on the first case our period is 3 on the second case the period is 36

The number 1.33333... must be written as $1.\hat{3}$ or $1.\overline{3}$ and 0.36363636... as $0.\overset{\frown}{36}$ or $0.\overline{36}$

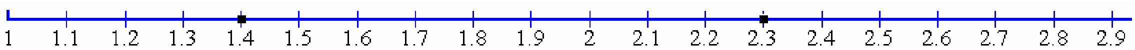
Irrational numbers: They have an unlimited quantity of decimal digits but there is not any period

Example: Calculating $\sqrt{2}$ we get 1.414213562… and we don't find any sequence on the digits we get.

3 Decimal numbers on the number line

Every decimal number has a place on the number line between two integers.

For example representing the numbers 2.3 or 1.4 we divide the units into ten equal parts and we find a point to represent these numbers



There is a peculiar propriety with the decimals on the number lien that is that between any two decimal numbers there is always a decimal number and it doesn't matter how close they are on the line.

Example between 2.4 and 2.5 we can find 2.44 and 2.4<2.44<2.5 or between 2.44 and 2.45 we can find 2.443 and 2.44<2.443<2.45, etc.

4 Rounding Decimal Numbers

Remember that to round a number to any decimal place value we look at the digit to the right of the place we wish to round to and when the digit 5, 6, 7, 8, or 9, appears in that place, you must add one unit to the last digit; when the digit 0,

1, 2, 3, or 4 appears in that place, you must cut the number.

Example 1.18 rounded to the nearest tenth is 1.2 while 3.453 rounded to the nearest hundredth is 3.45

Exercises

1. Write in words

3.006

45.204

34.40 metres

2. Order from less to high

a) 5.1, 4.99, 5.21, 5.201, $4.\widehat{9}$

b) –2, -1.78, 1.5, 1.8, 1.09

c) 0.0003, 0.1, 0.007, -1

3. Find two numbers between each pair of decimal numbers

a) 2.5 and 2.6 **b) 7.07 and 7.08**

c) –2.01 and –2 **d) 5.203 and 5.21**

e) - 0.001 and - 0.0009 **f) 78.00003 and 78.00005**

4. Plot on the number line the decimal numbers:

a) 1.3 b) 2.34 c) –2.5 d) 0.347 e) 79.45 f) –9.003

(Don't use the same number line for all the numbers!).

5 Round:

2.36 to the nearest tenth

6.3757 to the nearest hundredth

0.8903 to the nearest thousandth

17.17 to the nearest unit

176.705 to the nearest hundred

67,756 to the nearest thousand

5 Sexagesimal system

There are some magnitudes as angles and time in which the decimal system is not the only one that is used. Sexagesimal system is more frequently used; on this system every unit is divided into 60 equal parts to get the subunit.

Inclinometer

For the angles the unit is the degree.

The subunits of the degree are the **minute** and the **second**.

One minute $1' = \frac{1}{60}$ of a degree, that is $1^{\circ} = 60'$

One second $1'' = \frac{1}{60}$ of a minute, that is $1' = 60''$

Using this system an angle "a" can be expressed for example $a = 43^{\circ}\ 43'\ 20''$ and we need to operate angles expressed in this form.

For time the unit as you know is the hour and is divided into minutes and seconds with the same relationship they have on the angles.

A period of time is expressed as 3 h 5 min 3 s for example.

We need to be able to operate in this system using these two magnitudes.

Exercises

1. Convert into minutes

a) 7º **b) 21º**

c) 0.3º **d) 3º 4’ 55’’**

e) 5.67º

2. Convert into seconds

a) 2 min **b) 37 min**

c) 1 h 12 min **d) 2.5 h**

e) 3.47 min

3. Convert into hours

a) 345 min **b) 5 h 35 min**

c) 7 h 70 min **d) 5400 s**

4. Convert into degrees

a) 3700’’ **b) 23º 34’ 57’’**

c) 340’

5.1 Addition.

We need to add separately degrees or hours minutes and seconds and then convert the seconds into minutes and the minutes into degrees/hours if we get more than 60 subunits.

Example

Add $35^{\circ}\,23'\,30'' + 25^{\circ}\,53'\,58''$

Adding separately we get $35^{\circ}\,23'\,30'' + 25^{\circ}\,53'\,58'' = 60^{\circ}\,76'\,88''$ and as $88'' = 1'\,28''$ we add $1'$ and get $77' = 1^{\circ}\,17'$ we add 1º and the solution is $61^{\circ}\,17'\,28''$

5.2 Subtraction

We need to subtract separately degrees/hours minutes and seconds, if we do not have enough seconds or minutes we convert one degree/hour into minutes or a minute into seconds.

Example

Subtract 3 h 25 min 34 s and 1 h 46 min 50 s; we write 3 h 25 min 34 s as 2 h 84 min 94 s and

$$\begin{array}{r} 2\text{ h } 84\text{ min } 94\text{ s} \\ -\ \ 1\text{ h } 46\text{ min } 50\text{ s} \\ \hline 1\text{ h } 38\text{ min } 44\text{ s} \end{array}$$

5.3 Multiplication by a whole number

We multiply separately degrees/hours minutes and seconds and then convert the seconds into minutes and the minutes into degrees when we get more than 60 subunits.

Example

Multiply $(12^{\circ}\,33'\,25'')\cdot 4$

12º	33'	25"
4	4	4
48	133	100
	13'	$\boxed{1'}$ 40"
50º	$\boxed{2^{\circ}}$ 13'	

Solution 50º 20' 40"

5.4 Division by a whole number

We divide the degrees/hours, and the remainder is converted into minutes that must be added to the previous quantity that we had, divide the minutes and we repeat the same that we have done before. The remainder is in seconds.

Example

Divide $(34\text{ h }\ 25\text{min }\ 55\text{ s}):4$

34 h	13 min	25 s	4
2 h x 60 =	120 min		8 h 33 min 21" quotient
	133 min		
	1min x 60 =	60 s	
		85 s	
		1s remainder	

Exercises

1 Add:

a) 45º 55’ 31” + 56º 41’ 34”

b) 39 h 55 min 17 s+ 4 h 14 min 33 s

c) 233º 5’ 59” + 79º 48’ 40”

2 a) Subtract 56 h 24 min 16 s and 19 h 35 min 43 s

b) Calculate the complement of 33º 55’ 43”

c) Calculate the supplement of 102º 55’

3 Given A = 52º 12' 27" Calculate: a) 5·A b) 4·A c) $\frac{A}{3}$

4 We want to divide the full turn into seven equal sectors. Which is the angle of each sector?

5 A train arrives at a station at 17 h 35 min after a travel of 3 h 45 min, at what time did the train departed?

6 A sportsman starts its training at 8 h 43 min he runs go and back and spends 1 h 23 min 40 s going and 1 h 45 min 50 s coming back. At what time does it finish its training?

7 I spend 1 h 34 min going from A to B riding my bicycle. Running I need the double of the time and using my car I need the third. How long does it take to me going from A to B running and how long driving?

3 Fractions

Keywords			
Fraction	**numerator**	**denominator**	**simplest form**
improper fractions		**power**	**exponent**
index		**standard form**	**root**

Remember:

1 Fractions

A fraction is a number that expresses part of a unit or a part of a quantity.

Fractions are written in the form $\frac{a}{b}$ where a and b are whole numbers, and the number b is not 0.

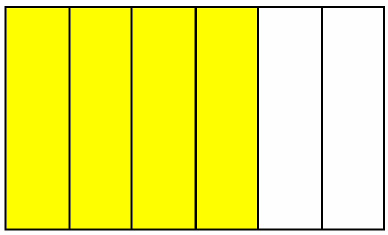
They can be written also in the form a/b

The number *a* is called the **numerator,** it is always an integer, and the number *b* is called the **denominator,** it can be any whole number except zero.

The **denominator** is the number, which indicates how many parts the unit is divided into.

The **numerator** of a fraction indicates how many equal parts of the unit are taken.

$\frac{4}{6}$ represents the shaded portion of the rectangle



2 Reading fractions

We use the cardinals to name the numerator and the ordinals for the denominator with two exceptions, when the denominators are 2 and 4, for denominators larger than 10 we can say "over" and do not use ordinal, so we read:

$\frac{1}{2}$ one half

$\frac{3}{2}$ three halves

$\frac{2}{3}$ two thirds

$\frac{1}{4}$ a quarter or a fourth

$\frac{12}{15}$ twelve over fifteen or twelve fifteenths

$\frac{6}{10}$ six tenths

$\frac{3}{4}$ three quarters or three fourths

$\frac{35}{126}$ seventeen over thirty-two

Exercise1 write in words the following fractions

$\frac{1}{3}$

$\frac{2}{5}$

$\frac{3}{7}$

$\frac{4}{10}$

$\frac{5}{12}$

$\frac{6}{19}$

$\frac{7}{200}$

$\frac{3}{10}$

$\frac{524}{1000}$

$\frac{3456}{5461}$

Exercise 2 express in figures:

Six sevenths **four elevenths** **A half**

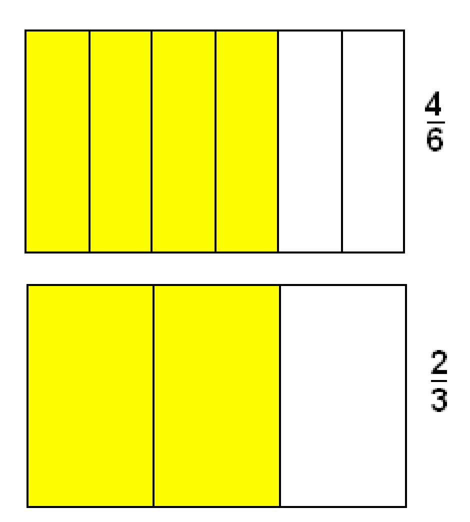
three quarters

seventeen over three hundred and forty one

thirty two over five hundred and twenty two

sixty two over seventy one

3 Equivalent fractions



Equivalent fractions are different fractions that name the same amount.

Example $\frac{4}{6}$ and $\frac{2}{3}$ are equivalent as can be seen in the drawing on the right

The rule is:

The value of a fraction does not change multiplying or dividing its numerator and denominator by the same number

The process of dividing numerator and denominator by the same number is called reduction

$\frac{12}{20}$ is equivalent to $\frac{3}{5}$, because we have divided both the numerator and the denominator by 4.

The fractions $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{100}{200}$ and $\frac{512}{1024}$ are all equivalent fractions.

We can test if two fractions are equivalent by cross-multiplying their numerators and denominators. This is also called taking the cross-product.

$$\frac{a}{b} \qquad \frac{c}{d}$$

So if we want to test if $\frac{12}{20}$ and $\frac{24}{40}$ are equivalent fractions

The first cross-product is the product of the first numerator and the second denominator: 12 × 40 = 480.

The second cross-product is the product of the second numerator and the first denominator: 24 × 20 = 480.

Since the cross-products are the same, the fractions are equivalent.

Simplest form

When numerator and denominator have no common factors the fraction is in the **simplest form** or in its **lowest terms**

We know that 4/12 = 2/6 = 1/3
4 and 12 have a common factor (4), so 4/12 can be written as 1/3 (Divide the top and the bottom by 4.)

2 and 6 have a common factor (2), so 2/6 can be written as 1/3 (Divide the top and the bottom by 2.)

However, 1 and 3 have no common factors, so **1/3 is the simplest form of these fractions.**

There are two methods of reducing a fraction to the lowest terms.

Method 1:
Divide the numerator and denominator by their HCF.

$\frac{12}{30}$. The HCF of 12 and 30 is 6 so $\frac{12}{30} = \frac{12:6}{30:6} = \frac{22}{5}$

Method 2:
Divide the numerator and denominator by any common factor. Keep dividing until there are no more common factors.

$$\frac{12}{30} = \frac{12:2}{30:2} = \frac{6}{15} = \frac{6:3}{15:3} = \frac{2}{5}$$

Exercise 3

a) Write a sequence of equivalent fractions as in the example of the first line

Start	Equivalent fractions					
$\frac{1}{2}$	$\frac{2}{4}$	$\frac{3}{6}$	$\frac{4}{8}$	$\frac{5}{10}$	$\frac{6}{12}$	$\frac{7}{14}$
$\frac{2}{5}$						
$\frac{3}{7}$						
$-\frac{1}{3}$						

b) Express these fractions in the simplest form

$\frac{20}{44}$ $\frac{26}{52}$

$\frac{29}{58}$ $\frac{70}{119}$

$\frac{84}{119}$ $\frac{42}{238}$

$\frac{20}{48}$ $\frac{56}{84}$

$\frac{22}{66}$ $\frac{34}{68}$

$\frac{52}{117}$ $\frac{54}{117}$

$\frac{39}{234}$ $\frac{20}{44}$

$\frac{72}{272}$ $\frac{324}{720}$

$\frac{720}{864}$ $\frac{272}{425}$

$\frac{510}{561}$ $\frac{3360}{3528}$

$\frac{3553}{5643}$ $\frac{7429}{8303}$

c) Express as its simplest form a fraction that describes these situations

1 I have 51 pens and 9 of them are black.

2 In our school 48 of the 84 teachers are women

3 Count how many boys and girls are in our class of 2º ESO and write down the fraction of each compared with the total number of pupils.

4 Comparing and ordering fractions

1. To compare fractions with the same denominator, look at their numerators. The largest fraction is the one with the largest numerator.

2. To compare fractions with different denominators, take the cross product. Compare the cross products.

   a. If the cross-products are equal, the fractions are equivalent.

   b. If the first cross product is larger, the first fraction is larger.

c. If the second cross product is larger, the second fraction is larger.

3. We can also compare fractions dividing the numerator by the denominator

5 Adding and subtracting fractions

1. If the fractions have the same denominator:

The numerator of the sum is found by simply adding the numerators over the denominator.

Their difference is the difference of the numerators over the denominator.

We do not add or subtract the denominators!

Reduce always when possible.

2. If the fractions have different denominators for example $\frac{3}{5}+\frac{7}{8}$

follow these steps:

1) Reduce them to a common denominator.

For this problem complete the following steps:

a) We find the LCM of the denominators as it is the smallest number that both denominators divide into. For 5 and 8, it is easy to see that 40 is the LCM. We look for equivalent fractions with 40 as common denominator.

In our example $\frac{3}{5}=\frac{[\ ]}{40}$ and $\frac{7}{8}=\frac{[\ ]}{40}$

b) We divide every new denominator by the previous one and then we multiply the result by the numerator.

So in $\frac{3}{5}=\frac{[\ ]}{40}$ we divide the new denominator 40 by the previous one that is 5, what gives us 8, we must multiply 3 by 8, so $\frac{3}{5}=\frac{24}{40}$.

Repeating the process with the second fraction:

$\frac{7}{8} = \frac{[\ ]}{40} = \frac{35}{40}$ We have multiplied the top and the bottom by 5

2) Add the numerators and do not change the denominator.

$$\frac{3}{5} + \frac{7}{8} = \frac{24}{40} + \frac{35}{40} = \frac{24+35}{40} = \frac{59}{40}$$

3) Reduce if possible.

Exercise 4

Calculate

a) $\frac{1}{2} - \frac{7}{9} + \frac{5}{6}$

b) $5 - \frac{3}{4} + \frac{17}{10}$

c) $\frac{2}{6} - \frac{5}{9} + \frac{8}{27}$

d) $\frac{5}{26} - \frac{7}{6} + \frac{25}{39}$

e) $3 - \frac{1}{3} + 1 - \frac{1}{5} - \frac{4}{15}$

f) $\frac{2}{5} - \frac{3}{10} - \frac{13}{35} + \frac{5}{14}$

Operate and reduce when necessary

a) $5-\left(\frac{7}{5}+\frac{4}{7}\right)-\frac{1}{35}$

b) $\frac{5}{3}+\frac{1}{2}-\left(\frac{3}{2}-\frac{1}{4}\right)+3$

c) $1-\left(\frac{3}{10}+\frac{11}{6}\right)$

d) $11-\left(\frac{7}{10}-\frac{2}{5}\right)$

e) $\left[\frac{3}{5}-\left(1-\frac{1}{3}\right)\right]-\left[2+\left(\frac{3}{5}+\frac{1}{9}\right)\right]$

f) $1+\left[\frac{3}{2}+\left(5-\frac{1}{4}\right)\right]-\left[3+\left(\frac{3}{5}+\frac{1}{10}\right)\right]-\frac{2}{15}$

g) $-\left[\frac{1}{5}-\left(2-\frac{1}{6}\right)\right]+\left[\frac{3}{10}-\left(\frac{2}{3}+\frac{1}{10}-\frac{7}{30}\right)\right]$

6 Improper fractions, mixed numbers

Improper fractions have numerators that are larger than or equal to their denominators.

For example $\frac{15}{7}$, $\frac{7}{7}$, and $\frac{18}{3}$ are improper fractions.

Mixed numbers have a whole number part and a fraction part.

For example $2\frac{3}{5}$, $5\frac{1}{3}$ are mixed numbers, meaning

$2\frac{3}{5} = 2 + \frac{3}{5}$ and $5\frac{1}{3} = 5 + \frac{1}{3}$.

Converting improper fractions to mixed numbers

To change an improper fraction into a mixed number, divide the numerator by the denominator. The quotient is the whole part and the remainder is the numerator of the fractional part.

For example

$$\frac{17}{3},\ 17:3 \Rightarrow \begin{cases} \text{quotient } 5 \\ \text{remainder } 2 \end{cases} \text{so } \frac{17}{3} = 5\frac{2}{3}$$

Converting mixed numbers to improper fractions

To change a mixed number into an improper fraction, multiply the whole number by the denominator and add it to the numerator of the fractional part.

For example

$$7\frac{2}{3} = \frac{7 \cdot 3 + 2}{3} = \frac{23}{3}$$

Note that converting mixed numbers to improper fractions is the same as adding whole numbers and fractions

Exercise 5

Convert the mixed numbers to improper fractions, operate and convert the results to mixed numbers.

a) $5\frac{1}{3} - \left(3\frac{2}{3} + \frac{4}{5}\right) + 4\frac{2}{15}$

b) $3\frac{5}{3}+\frac{13}{2}-\left(1\frac{3}{2}-\frac{1}{4}\right)+3\frac{5}{12}$

c) $7\frac{3}{9}+2\frac{1}{3}-\frac{12}{27}$

7 Multiplying fractions

When two fractions are multiplied, the result is a fraction with a numerator that is the product of the fraction's numerators and a denominator that is the product of the fraction's denominators. **Reduce when possible.**

Examples:

$\frac{7}{6}\cdot\frac{2}{5}=\frac{7\cdot 2}{6\cdot 5}=\frac{7}{3\cdot 5}=\frac{7}{15}$ We cancel the common factor of 2 in the top and bottom of the product. Remember that like factors in the numerator and denominator cancel out.

$$\frac{7}{6}\cdot\frac{2}{5}=\frac{7\cdot 2}{6\cdot 5}=\frac{7}{3\cdot 5}=\frac{7}{15}$$

Exercise 6

Operate and reduce when possible

a) $\frac{7}{3}\cdot\frac{1}{14}$

b) $\frac{3}{5}\cdot\frac{-5}{13}$

c) $-\frac{5}{6}\cdot\left(-\frac{2}{15}\right)$

d) $\frac{12}{5}\cdot 3\cdot\frac{5}{27}$

e) $\left(-\dfrac{26}{3}\right)\cdot\left(-\dfrac{1}{2}\right)\cdot\left(-\dfrac{9}{39}\right)$

f) $\left(1-\dfrac{4}{7}\right)\cdot\left(\dfrac{1}{3}+\dfrac{1}{2}\right)$

g) $\left(\dfrac{2}{7}-2\right)\cdot\left(1-\dfrac{5}{4}-\dfrac{25}{12}\right)$

8 Multiplying a fraction by a whole number, calculating a fraction of a quantity

To multiply a fraction by a whole number you must choose one of the two methods:

1. Write the whole number as an improper fraction with a denominator of 1, and then multiply as fractions.

2. Multiply the whole number by the numerator and do not change the denominator.

Example:

a) $6\cdot\dfrac{2}{7}=\dfrac{6}{1}\cdot\dfrac{2}{7}=\dfrac{12}{7}$

Exercise 7

Calculate:

a) $\dfrac{7}{12}$ **of 108**

b) $\dfrac{2}{7}$ **of 91**

c) $\dfrac{7}{8}$ **of 112**

d) $\frac{10}{23}$ **of 1311**

Calculate the unknown number in the following cases:

a) $\frac{3}{8}$ **of a number is 27, so the number is:**

b) $\frac{2}{3}$ **of a number is 64, so the number is:**

c) $\frac{17}{22}$ **of a number is 629 so the number is:**

d) $\frac{2}{17}$ **of a number is 360 so the number is:**

9 Dividing fractions

To divide fractions, multiply the first by the reciprocal of the second fraction.

The reciprocal of a fraction is obtained by switching its numerator and denominator.

We can also take the cross product.

To divide a number by a fraction, multiply the number by the reciprocal of the fraction.

Example:

$\frac{4}{5}:\frac{7}{11}=\frac{4}{5}\cdot\frac{11}{7}=\frac{44}{35}$ or simply taking the cross product $\frac{4}{5}:\frac{7}{11}=\frac{4\cdot 11}{5\cdot 7}=\frac{44}{35}$

Exercise 8

a) $\frac{5}{4}:\left(\frac{3}{2}+\frac{7}{4}\right)$

b) $\left(2-\frac{1}{12}\right):\left(\frac{1}{3}+\frac{1}{2}\right)$

c) $\left(\frac{2}{5}-\frac{1}{4}\right):\frac{11}{20}$

d) $\left(\frac{1}{3}-\frac{1}{5}\right):\left(\frac{1}{3}+\frac{1}{5}\right)$

e) $\frac{5}{3}\cdot\frac{1}{2}-\left(\frac{3}{2}:\frac{1}{4}\right):2$

f) $\left[\frac{2}{5}:\left(1-\frac{1}{5}\right)\right]\cdot\left[2-\left(\frac{3}{5}:\frac{9}{20}\right)\right]$

g) $\dfrac{1+\frac{1}{2}}{2-\frac{3}{4}}$

h) $\dfrac{\frac{1}{3}-\frac{3}{2}}{5+\frac{1}{3}}+\frac{7}{8}$

Exercises

9 There are 300 passengers on a train. At a station, $\frac{3}{5}$ of the passengers get off. How many people get off the train? How many people are left on the train?

10 Allan has 120€. He decides to save $\frac{2}{5}$ of this and to spend $\frac{1}{6}$ on books. How much does he save? How much does he spend on books? How much is left?

11 In a magazine there are three adverts on the same page. Advert 1 uses $\frac{1}{4}$ of the page, advert 2 uses $\frac{1}{8}$ and advert 3 uses $\frac{1}{16}$ of the page. What fraction of the page do the three adverts use?
An advert uses $\frac{3}{16}$ of the page, if the cost of an advert is 12€ for each $\frac{1}{32}$ of the page, How much does it cost?

12 A farmer owns 360 hectares of land. He plants potatoes on $\frac{3}{10}$ of his land and beans on $\frac{1}{6}$ of the remainder. How many hectares are planted with potatoes? How many hectares are planted with beans? How many hectares are left?

13 A journey is 120 miles. Richard has driven $\frac{3}{5}$ of this distance and in a second stage $\frac{5}{6}$ of the rest. How much farther does he have to drive to complete the journey?

14 Sue bought a record with $\frac{1}{4}$ of her money and she spent $\frac{1}{8}$ to see a movie. Which part of her money did she spend?

15 At a sale shirts are sold by $\frac{3}{5}$ of their original price and the sale price is 35€. What was the original price?

16 Joe spends $\frac{3}{8}$ of his salary on his own, gives $\frac{3}{5}$ of the remainder to his parents and saves 450€ what is his salary?

9 Powers

A power is a mathematical operation that indicates that many equal numbers are multiplied by themselves, it is written as a^n where ***a*** is the base and ***n*** is the exponent or index.

The meaning is: $a^n = \underbrace{a \times \cdots \times a}_{n},$

Examples:

$5^3 = 5x5x5 = 125$

$(-3)^4 = (-3)\cdot(-3)\cdot(-3)\cdot(-3) = +81$

$\left(\frac{2}{3}\right)^3 = \left(\frac{2}{3}\right)x\left(\frac{2}{3}\right)x\left(\frac{2}{3}\right) = \left(\frac{8}{27}\right)$

Remember how to name powers

5^4 Is read as

- **Five to the power of four**
- **The forth power of five**.
- **Five to the forth power**

We will read it as **five to the power of four**

There are two special cases with exponents 2 and 3 (squares and cubes), so we read, for example, 7^2 as **seven squared** and 6^3 as **six cubed.**

10 Rules for powers

1 Product

When doing the product of two powers **with the same base,** the base remains unchanged and the **exponents are added**

$$\boldsymbol{a^m \cdot a^n = a^{m+n}}$$

Example: $\mathbf{a^3 \cdot a^4 = (a \cdot a \cdot a)\cdot(a \cdot a \cdot a \cdot a) = a^{3+4} = a^7}$

Exercise 17 Fill in the missing numbers.

a) $3^3 \cdot 3^7 == 3^{[\ ]}$

b) $7^5 \cdot 7^8 = 7^{[\ \ ]}$ **c)** $[\ \ ]^2 \cdot 6^7 = 6^{[\ \ ]}$ **d)** $2^5 \cdot 2^{[\ \ ]} = 2^9$

e) $[\ ]\ \cdot [\ ]^4 = 2^5$ **f)** $5^3 \cdot 5^7 == 5^{[\ ]}$ **g)** $2^3 \cdot [\ \ \ ]^{[\ \ ]} = 2^9$

2 Divisions

In the quotient of two powers **with the same base,** the base remains unchanged and the **exponents are subtracted**

$$a^m : a^n = a^{m-n}$$

Example: $a^5 : a^2 = \dfrac{a \cdot a \cdot a \cdot \cancel{a} \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a}} = a^3$

Exercise 18 Fill in the missing numbers.

a) $\dfrac{7^{15}}{7^{12}} = 7^{[\quad]}$ **b)** $12^{13} : [\quad]^7 = 12^{[\quad]}$ **c)** $13^5 : [\quad]^3 = 13^2$

d) $3^{[\quad]} : [\quad]^2 = 3^7$ **e)** $\dfrac{3^{18}}{[\quad]^{13}} = [\quad]^{[\quad]}$ **f)** $[\quad]^2 : [\quad]^9 = 9^{[\quad]}$

3 Power of a product

The power of a product is the product of the powers $(a \cdot b)^n = a^n \cdot b^n$

Example: $(\mathbf{a} \cdot \mathbf{b})^4 = (\mathbf{a} \cdot \mathbf{b}) \cdot (\mathbf{a} \cdot \mathbf{b}) \cdot (\mathbf{a} \cdot \mathbf{b}) \cdot (\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a} \cdot \mathbf{a} \cdot \mathbf{a}) \cdot (\mathbf{b} \cdot \mathbf{b} \cdot \mathbf{b} \cdot \mathbf{b}) = \mathbf{a}^4 \cdot \mathbf{b}^4$

4 Power of a quotient

The power of a quotient is the quotient of the powers

$$(a : b)^n = a^n : b^n \text{ or } \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Example: $\left(\dfrac{\mathbf{a}}{\mathbf{b}}\right)^4 = \left(\dfrac{\mathbf{a}}{\mathbf{b}}\right) \cdot \left(\dfrac{\mathbf{a}}{\mathbf{b}}\right) \cdot \left(\dfrac{\mathbf{a}}{\mathbf{b}}\right) \cdot \left(\dfrac{\mathbf{a}}{\mathbf{b}}\right) = \dfrac{(\mathbf{a} \cdot \mathbf{a} \cdot \mathbf{a} \cdot \mathbf{a})}{(\mathbf{b} \cdot \mathbf{b} \cdot \mathbf{b} \cdot \mathbf{b})} = \dfrac{\mathbf{a}^4}{\mathbf{b}^4}$

5 Power of a power

When powering another power we multiply the exponents

$$(a^m)^n = a^{m \cdot n}$$

Example: $\left(\mathbf{a^4}\right)^{\mathbf{3}} = \mathbf{a^4} \cdot \mathbf{a^4} \cdot \mathbf{a^4} = \mathbf{a^{4+4+4}} = \mathbf{a^{4\cdot 3}} = \mathbf{a^{12}}$

Exercise 19 Fill in the missing numbers.

a) $\left(4^2\right)^5 = 4^{[\ \ ]} = 2^{[\ \ ]}$ **b)** $\left((-3)^2\right)^{[\ \ ]} = 3^8$ **c)** $\left(3^{[\ \ ]}\right)^2 = 3^8$

d) $\left([\ \ \ ]^2\right)^3 = 5^6$ **e)** $\left(2^2\right)^{[\ \ ]} = [\ \ \ ]^8$

6 Negative exponents

$$\boldsymbol{a^{-n} = \frac{1}{a^n}}$$

Example: $\mathbf{2^{-3}}$ means $\mathbf{\frac{1}{2^3} = \frac{1}{8}}$

Exercise 20 Fill in the missing numbers.

a) $(\ \ )^{-3} = \frac{1}{27}$ **b)** $(7)^{-3} = \frac{1}{7^{[\ \ ]}} = \frac{1}{[\ \ \ ]}$

c) $\left([\ \ \ ]^{-2}\right)^{-3} = 5^6$ **d)** $\left(2^2\right)^{[\ \ ]} = [\ \ \ ]^{-8}$

7 Scientific notation, standard form

There is a way of writing numbers that are too large and also for numbers that are too small using powers of ten. Big numbers are written in the form: $a \times 10^b$, where b is a whole and positive number.

Example $70\ 000 = 7x10^4$ and $237\ 000\ 000 = 2.37x10^8$

Small numbers can be written as $a \times 10^b$, where b is negative.

Example: $0.0037 = \frac{3.7}{100000} = \frac{3.7}{10^5} = 3.7x10^{-5}$

Exercise 21 Express in standard form the following numbers:

a) 34,000,000,000

b) 3.5 billions

c) 357,650,000

d) 0.034

e) 0.000000056

f) The number of seconds in a year (round appropriately)

g) The length of your class room in km

h) Calculate how many litres of water there are in a swimming pool that measures 25m length, 8.5m width and 2 m depth

Exercise 22 Write using the ordinary decimal notation

a) 5.43×10^{8} **b)** 1.05×10^{2} **c)** 2.055×10^{13}

d) $7.26\text{x}10^{-2}$ **e)** $7\text{x}10^{-7}$ **f)** $90000\text{x}10^{-7}$

Exercise 23 Express as just one power

a) $7^3 \cdot 7^{-5} =$ **b)** $4^{12} \cdot 4^{-3} \cdot 4^6 \cdot 4 =$

c) $9^2 \cdot 9^7 \cdot 9^{-2} =$ **d)** $5^7 : 5^3 =$

e) $\dfrac{5^7}{5^{-3}} =$ **f)** $\left(14^2\right)^4 =$

g) $\dfrac{13}{13^2 \cdot 13^3} =$ **h)** $\left(5^2 \cdot 3^2\right) = [\quad]^2$

i) $\left(11^2 \cdot 11^3\right)^6 =$ **j)** $\left(\dfrac{6^2}{3^2}\right) = [\quad]^2$

k) $\left[3^8 : \left(3^{-2} \cdot 3^3\right)\right]^2 =$ **l)** $\dfrac{5^7 \cdot 5^3}{\left(5^4\right)^2} =$

Exercise 24 Calculate

a) 5^0 **b)** $(-1)^3$ **c)** $(-2)^5$ **d)** $(-1)^{14}$

e) $(-3)^4$ **f)** -3^4 **g)** 2^{-3} **h)** 10^{-5}

i) 7^{-2} **j)** $\left(2^{-3}\right)^3$

8 Fractional indices roots

A power with a fractional exponent means

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Examples:

$$\mathbf{a^{\frac{1}{2}} = \sqrt{a}}$$
$$\mathbf{a^{\frac{1}{3}} = \sqrt[3]{a}}$$
$$\mathbf{a^{\frac{1}{4}} = \sqrt[4]{a}}$$
$$\mathbf{a^{\frac{1}{n}} = \sqrt[n]{a}}$$

Exercise 25 Convert into a root and calculate the value of the following:

a) $100^{\frac{1}{2}}$ **b)** $0.001^{\frac{1}{3}}$

c) $\left(27^{\frac{1}{3}}\right)^2$ **d)** $27^{\frac{2}{3}}$

4 Proportions

Keywords

Ratio **proportion** **mean** **extreme**

1 Ratio

A **ratio** is like a fraction. If we want to compare two quantities we can divide both numbers, then we can express it

a) As a fraction

b) As a decimal number

c) As a ratio

Example 1 Comparing the numbers 6 and 2, we write $\frac{6}{2}=3$ which means that 6 contains the number 2 three times.

Comparing now the numbers 5 and 2

We can write the comparison as:

a) A fraction $\frac{5}{2}$

b) A decimal number 2.5

c) A ratio 5:2 (read “5 to 2”)

The three possibilities have the same meaning, the difference between a ratio and a fraction is that in fractions we only use whole numbers and in ratios we can also use decimal numbers.

As fractions and ratios mean the same thing, the properties of the fractions can also be used in ratios, one important property is:

We can multiply or divide both terms in a ratio by the same number.

A ratio in its most simple form is the fraction in its lowest terms (always expressed with whole numbers).

Example 2 These ratios are equivalent

12:8 = 3:2 (3:2 is the ratio expressed in its simplest form)

4.5:9 = 9:18 = 1:2. (1:2 is the ratio expressed in its simplest form).

Exercise 1 Express the following ratios in their most simple form

a) 16 to 8

b) 8 to 20

c) 4.5:5

d) 4.5:3

e) 3/7:2

f) 15:21

2 Proportions

When two ratios are equal, the four terms are in **proportion** or are a proportion, so $\frac{3}{7}=\frac{6}{14}$ or 3:7 = 6:14 (is read "three is to seven as six is to fourteen) are a proportion.

If we write $\frac{a}{b}=\frac{c}{d}$

b and c are called the **means**

a and d are called the **extremes**

A fundamental property of proportions is that **the product of the extremes is equal to the product of the means**; this is very useful when we know three of the numbers of a proportion and want to calculate the fourth.

If we have $\frac{a}{b} = \frac{x}{c}$, we say $b \cdot x = a \cdot c$ and then $x = \frac{a \cdot c}{b}$.

The rule is **one mean is the product of the extremes divided by the other mean.**

If we have $\frac{a}{x} = \frac{b}{c}$ and $x = \frac{a \cdot c}{b}$

The rule is **one extreme is the product of the means divided by the other extreme.**

Exercise 2 Write three different proportions with these ratios:

a) 3:2

b) $\frac{7}{5}$

c) 2.3:5

d) 7.2 to $1\frac{3}{4}$

Exercise 3 Calculate the unknown number in these proportions

a) $\frac{2}{3} = \frac{5}{x}$

b) $\frac{5}{2.4} = \frac{x}{8}$

c) $\frac{x}{9} = \frac{12}{24}$

d) $\frac{2}{x} = \frac{9}{2}$

Exercise 4

a) Which number has the same ratio to 5 as 4 to 9?

b) Find x if $\frac{4}{12} = \frac{12}{x}$ **(x is called third proportional to 4 and x)**

c) Find x if $\frac{18}{5} = \frac{7}{x}$ **(x is the fourth proportional to 18, 5 and 7)**

d) Find x if $\frac{4}{x} = \frac{x}{5}$

3 Direct proportions

We say that there is a direct proportionality between two magnitudes if an increase on one magnitude causes a proportional increase on the other and a decrease on the first quantity causes a proportional decrease on the second.

Note that a direct proportion is the same as proportions we have seen on the previous point.

A direct proportion is also called simple proportion.

The best way to recognize if two magnitudes are in a direct proportion is to see if when we double one the other also doubles and if we half the first, the other also halves.

When we are trying to find a number in an exercise involving magnitudes in a direct proportion there are two methods:

I -Unitary method

1. We convert the proportion in 1:n or n:1 (the most convenient)

2. We multiply by the third quantity.

Example 3: A man walks 5200m in $2\frac{1}{2}$ hours. How much will he walk in 7 h at the same speed?

1. If he walks 5200 m in 2.5 h, in 1 h he will walk $\frac{5200}{2.5} = 2080\,\text{m}$.
2. In 7 h he will walk $2080 \cdot 7 = 14560\ \text{m}$

II –The fractional method or proportion.

You must work as we have seen before with proportions.

Example: With the same data as in example 3:

$$\frac{5200\ \text{m}}{2.5\ \text{h}} = \frac{x\ \text{m}}{7\ \text{h}} \Rightarrow x = \frac{5200 \cdot 7}{2.5} = 14560\ \text{m}$$

It is a good idea to write always the units so we can see that we are organising the quantities correctly.

Exercise 5 Complete the table with the cost in pounds of a piece of silk

Metres of silk	3	7		12	1
Cost in £	5		9		

Exercise 6 Complete the table if the magnitudes are directly proportional

Metres of silk	1		4	2	
Cost in £		5	50		7

Exercise 7 Complete the table of the cost in € and the litres of petrol bought

Petrol in litres	1		5	7	
Cost in €		40	6.20		50

Exercise 8 A phone call costs €0.25 each 2 minutes or fraction rounded to the seconds. Complete the table.

Call length in min.	7.5		13min 25 s
Cost in €		12	

Exercise 9 Richard earns £17.5 for working 7 hours. How much will he earn for working 9 hours.

Exercise 10 We have paid for 7 nights in "Hotel los Llanos" 364€. How much will we pay for 3 nights? How much for 15 nights?

Exercise 11 For cooking a cake for 6 people the recipe says that we need 3 eggs, 150g of flour and 50 g of sugar. Calculate how much of each ingredient we need to cook a cake for 9 people.

Exercise 12 Mara has exchanged \$370 and has received 255€ without any commission. How much in € will she receive for \$20? How much in \$ will she receive for 60€?

Exercise 13 A 25 kg tin of paint covers 70 m^2 of wall. How many kg would be needed to cover 53 m^2 of wall?

Exercise 14 My car uses 16 litres of petrol to travel 250 km.

a) How far can I travel with 55 litres?

b) How much petrol would I need to travel 180 km?

4 Inverse proportions

We say that there is an inverse proportionality between two magnitudes if an increase in one magnitude causes a proportional decrease in the other and a decrease in the first magnitude causes a proportional

increase in the other. That is if one magnitude is multiplied by 2, 3, … this causes in the second a division by 2, 3, … etc.

Example 4: If 18 men can do a job in 10 days, in how many days will 45 men do the same job?

This is an inverse proportion because with double the men, half the days are required. We will work in two steps.

1. Write the proportion (be careful! The same magnitude on each side) $\frac{18\text{ men}}{45\text{ men}} = \frac{10\text{ days}}{x\text{ days}}$
2. Make the inverse in one ratio $\frac{18\text{ men}}{45\text{ men}} = \frac{x\text{ days}}{10\text{ days}}$
3. Solve as we have done previously $x = \frac{18 \cdot 10}{45} = 4\text{ days}$.

Exercise 15 Two pumps take 5 days to empty a pool. How long will 5 pumps take to empty the same pool?

Exercise 16 At 65 km/h a journey takes 5 h 25 min. how long will the journey take at a speed of 75 km/h?

Exercise 17 It takes 12 hours for 3 bricklayers to build a wall. How long will it take for 5 bricklayers?

Exercise 18 At 130 km/h a train takes 1h and 23 min for its journey from Albacete to Valencia. How long will the same journey take with the AVE at a speed of 190km/h? How long with a speed of 250 km/h?

Exercise 19 A company needs 33 workers to pack its production in 25 days, if the total production needs to be packed in 15 days. How many extra workers do they need?

Exercise 20 James can write 8 pages with 25 lines per page in one hour. How many pages can he write if there are 20 lines on each page?

Exercise 21 If 2 eggs take 6 minutes to boil, how long will 5 eggs take?

Exercise 22 There have been 13 winners in a lottery and each one will receive 23000€, but there are 3 more winners, how much will each one receive now?

Exercise 23 If i ride my bicycle at an average speed of 15 km/h I travel a distance of 22 km in a certain period of time, if the speed is 17 km/h, how far will I travel?

Exercise 24 Crash barriers are to be put on a stretch of motorway. We need 56 pieces of material which are each 2.5 m long. How many pieces of 3.5 m will we need for the same stretch?

5 Percentages

Keywords			
Percentage	**percent**	**increase**	**decrease**
Interest	**principal**	**rate interest**	**mixture**
proportional division			

1 Percentage

A **percent** is a ratio to 100.

A **percentage** can be considered as a fraction with denominator 100.

Percent and hundredths are basically equivalent. This makes conversion between percent and decimals very easy.

To convert from a decimal to a percent, just move the decimal 2 places to the right.

To convert a fraction into a percentage we multiply the numerator by 100 and we make the division.

There are some very easy cases of percentages that we can match mentally to fractions such as $50\% = \frac{1}{2}$, $25\% = \frac{1}{4}$, $20\% = \frac{1}{5}$, $10\% = \frac{1}{10}$, etc.

Example 1

a) The ratio 3:8 or the fraction $\frac{3}{8}$ can be expressed as $\frac{3 \cdot 100}{8} = 37.5\%$.

b) $\frac{10}{5} = \frac{10 \cdot 100}{5} = 200\%$

Summarising ideas we can write:

As a proportion or a fraction	As a decimal number	As a percentage
3:8	0.375	37.5%
$\frac{10}{5}$	2	200%
Making the division $\Rightarrow$	Multiplying by 100 $\Rightarrow$	

Exercise 1 Complete the table doing the calculations in your notebook.

Proportion or fraction	**Decimal number**	**Percentage**
		30%
		70%
	0.27	
	1.5	
$^{2}/_{7}$		
$^{35}/_{12}$		
	0.68	
		28.3%
2.8:5		

Exercise 2 Convert the following percentages to fractions.

a) 35% **b) 3.4%**

c) 560% **d) 40%**

e) 45% **f) 0.7%**

g) 15% **h) 0.02%**

Exercise 3 Convert the following to percentages.

a) 0.15 **b) 0.634**

c) $\frac{1}{6}$ **d)** $\frac{5}{12}$

e) 0.12 **f) 1.34**

g) 0.06 **h) 0.75**

i) $\frac{3}{5}$ **j)** $\frac{1}{16}$

2 Calculating a percentage of a quantity

We know that for calculating a fraction of a quantity we must multiply the fraction by the number

Example 2 $\frac{5}{8}$ of 200 is $\frac{5}{8} \cdot 200 = 125$ it is the same with percentages

To calculate the percentage of a quantity we must multiply it by the percent and divide by 100.

Example 3

Calculate the 35% of 28 $= \frac{35}{100} \cdot 28 = 9.8$

Exercise 4 Calculate

a) 25% of 68 **b) 32.5% of 500**

c) 70% of 25 h 45 min **d) 1% & 3% of 50**

e) 1% & 5% of 600 **f) 1%, 0.5% & 6% of 500**

g) $5\frac{1}{2}\%$ **of 800** **h) 15% of 5€**

i) 7% of 120$

Exercise 5 The population of a town is 652000 and 35% of them live in the centre district. How many of them live in this district?

Exercise 6 You have the possibility of choosing a prize:

Prize 1: 25% of 570€. Prize 2: 30% of 450€. Prize 3: 95% of 150€

Which one do you choose? Explain your answer.

3. Calculate the total from the percent.

Calculating the total from the percent and the part can be done using direct proportion.

Example 4 In the class 13 students didn’t do their homework; this was 52% of the class. How many students are in this class?

We can say 52 ------------ 100 total

13 ------------ x total so $x = \frac{13 \cdot 100}{52} = 25$ students.

Exercise 7 In a sale the price of a television set is 150€ which is 65% of the usual price, what was the original price?

Exercise 8 The 6% of the population of Albacete are immigrants and there are 9900 immigrants living in our city, what is the population of Albacete?

4 Percentage increase decrease

4.1 Calculate a number increased or decreased in a percentage

It can be done in two steps:

1. Calculate the % of the quantity
2. Add or subtract the percentage to the original quantity.

Example 5 the population of a town is 63500 and last year it increased by 8%, what is the population now?

1. 8% of 63500 are $\frac{63500 \cdot 8}{100} = 5080$
2. The current population is 63500 + 5080 = 68580 people

We can also use a formula, if a is the % increase, **c** the initial quantity and **f** the final quantity, then: $f = \left(1 + \frac{a}{100}\right) \cdot c$, in the same example, we can say $f = \left(1 + \frac{8}{100}\right) \cdot 63500 = 1.08 \cdot 63500 = 68580 \text{ people}$.

Example 6 The price of some clothes is 68€ and there is a discount of 7%, what is the final price?

1. The discount is 7% of 68, $\frac{68 \cdot 7}{100} = 4.76€$
2. The final price is 68 – 4.76 = 63.24€

We can use a similar formula, being **a** is the % decrease, **c** the initial quantity and **f** the final quantity, $f = \left(1 - \frac{a}{100}\right) \cdot c$.

4.2 Finding the original amount

If we know the % increase or decrease and the value we can find the total using proportions.

Example 7 The net salary of an employee is 1230€ after paying 18% of IRPF, what is the gross of his/her salary?

82€ net --------- 100€ gross

1230€ net --------- x€ gross and $x = \dfrac{1230 \cdot 100}{82} \approx 1500€$

Example 8 I have bought a pair of jeans for €32, the IVA is 16%, what was the price before IVA?

100€ net --------- 116€ plus IVA

x € net ----------- 32€ plus IVA and $x = \dfrac{32 \cdot 100}{116} \approx 28€$

Exercise 9 For each quantity including 12% IVA calculate the original cost excluding IVA.

a) 147.84 €

b) 65€

c) $2072

d) 500€

Exercise 10 Calculate the price of the following items before 15% discount.

a) Jacket £200

b) Car £15800

c) Shoes £80

Exercise 11 The price of an electric oven before taxes is 560€ plus 17% IVA and the salesman offers a 12% discount, what is the final price?

Exercise 12 Find the current price for each item.

Item	Car	Coat	Book	Bus ticket	Holliday	Globes	House
Old price	25.000€	350€	58€	2.5€	1800€	25€	250.000€
Change	7% incr.	30% disc.	12% disc.	12% incr.	6% incr.	25% disc.	3% incr.

4.3 Finding the % increase or decrease

If we know the amount of the increase or decrease and the initial value we can find the % increase or decrease using proportions.

Initial value ----------- increase/decrease

100 -------------------- x % increase/decrease

A formula can be used:

$$\%\text{increase} = \frac{\text{increase}}{\text{initial value}} \cdot 100 \text{ or } \%\text{decrease} = \frac{\text{decrease}}{\text{initial value}} \cdot 100$$

Example 9 Last year there were 1560 employees in a company, this year 230 new people have been employed. What has been the % increase of the staff in the company?

$$\%\text{increase} = \frac{230}{1560} \cdot 100 = 14.7\%$$

Example 10 I have paid 215€ for a coat and the original price was 230€. What is the % discount?

$$\%\text{discount} = \frac{230 - 215}{230} \cdot 100 = 6.52\%$$

5. Interest

When money is invested in a bank or any other financial society or in the case of a loan each year interest is directly proportional to the amount of the deposit or the loan.

If we consider that the annual interest is not added to the **principal** (original amount), this is simple interest in this case the formula for the interest or benefit after t years is:

$$I = \frac{C \cdot r \cdot t}{100}$$

Where: **I** is the total interest, **C** is the **principal**, **r** is the % **rate interest** and **t** is the number of periods, usually years.

Example 11 An amount of 1200€ at 4% per year produces after 5 years

$$I = \frac{1200 \cdot 4 \cdot 5}{100} = 240€$$

We can use also proportions and work in to steps

For one year 100€ principal -------- 4€ interest

1200€ principal -------- x€ interest

So $x = \frac{1200 \cdot 4}{100} = 48€$

And in 5 years $I = 48 \cdot 5 = 240€$

But you can see that it is not advisable.

Exercise 13 Calculate the interest for a deposit of 2450€ in the following cases:

a) At 4% in 5 years

b) at 6% in 4 years

c) at 12% in 12 year

Exercise 14 Complete the table

Principal €	**300**	**2300**			**5000**
Annual rate	**4.3%**	**5%**	**2%**	**4%**	
Nº of years	**7**	**16**	**1**	**3**	**5**
Interest €			**130**	**500**	**1125**

Calculations here

6. Mixtures

These exercises are about how to find the price of a mixture of several quantities of products with different prices in order not to have any profit or loss when they are sold or, in other occasions, having a certain benefit.

Example 12 We mix 34 kg of tea at 2.5€ per kg with 25 kg of tea at 3.5€ per kg and 71 kg of tea at 4.5€ per kg. At what price per kg must we sell the mixture?

We must calculate the total price of the whole tea and divide it by the number of kg of tea; it is very useful to organize the calculations as follows:

Item	weight	price per kg	total cost
Tea 1	34kg	2.5€	85€
Tea 2	25kg	3.5€	87.5€
Tea 3	71kg	4.5€	319.5€
Mixture	130kg	x €	492€

So each kg of the mixture must be sold at $\frac{492}{130} = 3.78€$ per kg

Example 13 A mixture consists of 15 kg of coffee purchased at 8€/kg and chicory purchased at 2€/kg, how many kg of chicory do we need to mix if we want to sell the mixture at 6€/kg?

We organize our calculations in a similar way

Item	difference of price	nº of kg	total
Coffee	8 – 6 = 2€	15	30€ loos
Chicory	6 – 2 = 4€	x	4x€ gain

As the loss must be equal to the gain $4x = 30 \Rightarrow x = \frac{30}{4} = 7.5\text{kg}$

Exercise 14 A merchant mixes 50l of spirit at 4€/l with 40l at 3€/l and 10l of water costing nothing, he wants to have a profit of 12%, what price should each l of the mixture be?

Exercise 15 12l of cologne at 25€/l have been mixed with 17l at 12€/l if the mixture is sold at 18€/l calculate the % of gain or loss. Sol: 3.5% gain

Exercise 16 How many kg of coffee at 7.5€/kg do we need to mix with 12kg of coffee at 10€/kg so the mixture can be sold at 8€ per kg without any gain or loss? Sol: 48Kg

7. Proportional division

In this kind of problems something has to be shared in proportion to certain data.

It is very common in business; for example, that two or more people form a partnership and they each invest a different amount of money, then the profit is divided among them in proportion to the invested capital.

Example 14 Three partners invest 30,000, 40,000 and 50,000€ respectively in a business. After a year the profit is 7500€, how much each one of them receive?

The total invest is 30,000 + 40,000 + 50000 = 120,000€

So the profit for each one is:

$B_1 = \dfrac{7500 \cdot 30000}{120000} = 1875€$, $B_2 = \dfrac{7500 \cdot 40000}{120000} = 2500€$ and $B_3 = \dfrac{7500 \cdot 50000}{120000} = 3125€$

Final exercises

Exercise 17 Convert the following percentages to fractions giving the answer in the simplest form.

a) 12% **b) 15%**

c) 82% **d) 17%**

e) 120% **f) 50%**

g) 45% **h) 80%**

i) 60% **j) 78%**

k) 4% **l) 175%**

Exercise 18 Convert the following decimals to percentages.

a) 0.17 **b) 0.03**

c) 0.15 **d) 0.9**

e) 1.5 **f) 3.12**

Exercise 19 Convert the following fractions to percentages

a) $\frac{2}{5}$ **b)** $\frac{4}{7}$

c) $\frac{4}{7}$ **d)** $\frac{5}{10}$

e) $\frac{3}{4}$ **f)** $\frac{1}{3}$

g) $\frac{6}{25}$ **h)** $\frac{13}{50}$

i) $\frac{8}{9}$ **j)** $\frac{7}{8}$

Exercise 20 Calculate using mental method when appropriate

a) 12% of 515 **b) 50% of 800**

c) 7% of 3570 **d) 112% of 300**

e) 25% of 1000 **f) 13.4% of 27**

g) 5% of 80 **h) 90% of 180**

Exercise 21 Express each of the following as percentages.

a) 17 out of 50 **b) 5 out of 15**

c) 6 out of 35 **d) 2 out of 20**

e) 7 out of 155 **f) 67 out of 130**

g) 86 out of 600 **h) 70 out of 200**

Exercise 22 We make a lemon refresh with 230ml of pure lemon juice and 340ml of water, what is the percentage of lemon in this beverage?

Exercise 23 A farmer sells 420 sheep of its 700 in total. What percentage of sheep has he sold?

Exercise 24 A boy spends 6 euros of his pocket money on the cinema, 7.5€ on a sandwich and a coke and he still has 8€ left.
Which percentage of the whole has he left?
Which percentage has he spent on each thing?

Exercise 25 Joe's mother has cooked 40 cookies; he has eaten 7, which percentage of the cookies is left?

Exercise 26 In our school there are 990 students and 91 teachers, which is the percentage of teachers compared with the number of students?

Exercise 27 The value of a house is 120,000€ and the value of the content is 25,000€ express the content value as a percentage of the total.

Exercise 28 An 80ml bottle of perfume contains 25ml of extra free perfume, what is the percentage of free perfume?

Exercise 29 A beverage contains 300ml of orange juice, 250ml of water and 300ml of milk. Express the percentage of each component of the drink.

Exercise 30 Calculate the final amount:

a) £1500 plus 7.5% VAT

b) 2700€ plus 12% IVA

c) 3200€ of salary increased in 2.3%

d) 12€ with a 17% discount

e) 1530€ after an 8% discount

f) £170 reduced by 15%

Exercise 31 Peter earns 1200€ per month, but from this amount he needs to pay 21% of IRPF, how much does he take home?

Exercise 32 The price of a bicycle is 230€ plus 6% IVA and there is a special offer of 12% discount, which is the final price of the bicycle?

Exercise 33 I have paid 788.5€ for a computer in an offer of 17% discount, which was the usual price of the computer?

Exercise 34 Only 9000 of the 20,000 different types of fish are caught, what percentage of the types of fish do fishermen catch?

Exercise 35 In a shop there is an offer as listed

ITEM	SALE PRICE	DISCOUNT
LCD TV SET	509€	12%
COMPUTER	644€	5%
DVD	315€	10%
PDA	164€	18%

Which were the usual prices of these items?

Exercise 36 Three farmers have sold 350, 200 and 170 kg of pears to a salesman and he has paid 1080€ in total. How much will each farmer receive?

Exercise 37 A man gives 180€ to his three children aged 3, 5 and 8 years old, they must share the money proportionally to their ages. How much does each one get?

Exercise 38 Divide 195 into three parts proportional to:

a) **2, 3 and 4**

b) $\frac{1}{2}, \frac{1}{3}$, **and** $\frac{1}{4}$

Exercise 39 A merchant of spirits mixes 50l of wine at 1.20€/kg with 120kg of wine at 1.8l/kg, what is the price of each l of the mixture?

Exercise 40 A grocer mixes two different types of oranges, 65kg costing 3€/kg with 25kg costing 4€/kg, he sells the oranges at 3.5€/kg what is his gain percent? What should be the price of each kg so not to have any gain or loss?

Exercise 41 How many kg of potatoes at 1.8€/kg must be mixed with 50kg of potatoes at 2.5€/kg so that the potatoes could be sold at 2€per kg?

6 Algebra

Keywords				
Algebra	**equation**	**expression**	**formula**	**identity**
Monomial	**coefficient**	**degree**	**variable**	**like term**
polynomial				

1 Using letters for numbers

Algebra is a branch of mathematics in which symbols, usually letters of the alphabet, represent numbers or members of a specified set and are used to represent quantities so that we can use letters for the arithmetical operations such as +, –, ×, ÷ and the power.

What do you do when you want to refer to a number that you do not know? Suppose you wanted to refer to the number of buildings in your town, but haven't counted them yet. You could say 'blank' number of buildings, or perhaps '?' number of buildings.

In mathematics, letters are often used to represent numbers **that we do not know** - so you could say 'x' number of buildings, or 'q' number of buildings. **These are called variables**.

Look at these examples:

- The triple of a number: $3n$
- The triple of a number minus five units: $3n - 5$
- The following number to x: $x + 1$
- The preceding number to y: $y - 1$
- An even number: $2a$
- An odd number: $2z + 1$ or $2z - 1$

Exercise 1

Find the expression:

1. **I start with x, double it and then subtract 6.**

2. **I start with x, add 4 and then square the result.**

3. **I start with x, take away 5, double the result and then divide by 3.**

4. **I start with x, multiply by 4 and then subtract t.**

5. **I start with x, add y and then double the result.**

6. **I start with a, double it and then add b.**

7. **I start with n, square it and then subtract n.**

8. **I start with x, add 2 and then square the result.**

9. **A brick weighs x kg. How much do 6 bricks weigh? How much do n bricks weigh?**

10. **A man shares x euros between n children. How much does each child receive?**

2. Mathematical language.

In Algebra, you use letter symbols to represent unknowns in a variety of situations:

EQUATIONS	In an **equation** the letters stand for one or more particular numbers (the solutions of the equation), for example, $2x + 1 = x - 2$
EXPRESSIONS	In an **expression** there is no equals sign, for example, $3x^2 + 2x - 1$.
IDENTITIES	In an **identity** there is an equals sign (sometimes written ≡) but the equality holds for *all values* of the unknown, for example: $2(x + 1) = 2x + 2$
FORMULAE	In a **formula** letters stand for defined quantities or variables, for example, $d = s \cdot t$. (d is distance, s is speed and t is time)

Exercise 2

Separate the equations, the formulae, the identities and the expressions:

a) $x (x + 1) = x^2 + x$

b) $7y + 10$ c) $V = I \cdot R$

d) $x^2 - 3x + 10$

e) $7x + 11 = x - 9$

f) $(x + 1)^2 = x^2 + 2x + 1$

g) $A = \pi \cdot r^2$

h) $x^2 - 7x = 0$

Exercise 3

Write an equation for the following statements:

a. **If you multiply a number by 3 and then add 4, the answer is 13**

b. **The addition of a number and its consecutive is 81.**

c. **If you multiply the number by 2 and then subtract 5, the answer is 4.**

d. **If you subtract 11 from the number and then treble the result, the answer is 20.**

e. **If you treble the number, take away 6 and then multiply the result by 2, the answer is 18.**

f. **If you multiply the number by 5 and subtract 4, you get the same answer as when you add 3 to the number and then double the result.**

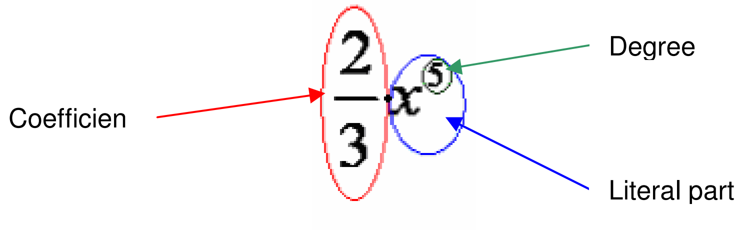
g. **The sum of four consecutive numbers is 90.**

h. **The sum of three consecutive odd numbers is 177.**

i. **When a number is doubled and then added to 13, the result is 38.**

3. Monomials.

A monomial is an algebraic expression consisting of only one term, which has a known value (coefficient) multiplied by one or some unknown values represented by letters with exponents that must be constant and positive whole numbers (literal part). For example:



If the literal part of a monomial has only one letter, then the degree is the exponent of the letter.

If the literal part of a monomial has more than one letter, then the degree is the sum of the exponents of every letter. For example:

- The degree of $-2x^3$ is 3
- The degree of $3x^5y^2$ is 5 + 2 = 7.
- The degree of $7x^3yz$ is 3 + 1 + 1 = 5.

Exercise 4 Complete the table

Monomial	Variables	Coefficient	Literal part	degree
$-3x^2y^3$	x and y	-3	x^2y^3	5
$7x^3y\ z$				
$-\frac{4}{5}x^3$				
3^2x^3				
5				

Exercise 5 Name the variables, coefficient, literal part and degree of the following monomials

a) $5xy$

b) $-2x$

c) $5x^{-2}y^{3}$

d) xy^{2}

e) $-x^{2}yz$

f) $-2\sqrt[3]{x}$ g) $\sqrt[5]{7}x^{5}y^{8}$

h) $\frac{3}{x}$

j) $-\frac{3}{\sqrt{7}}x^{4}y^{2}z^{4}t$

k) 8

l) $0x^{2}y$

4. Addition and subtraction of monomials.

You can add monomials only if they have the same literal part (they are also called **like terms**). In this case, you sum the coefficients and leave the same literal part.

Like terms use **exactly the same literal part**
3x and 6x are **like** terms
3x and 3xy are **unlike** terms
3x and $3x^{2}$ are **unlike** terms

Look at these examples:

$$4xy^{2}+3xy^{2}=7xy^{2}$$

$5x^2 + 3 - 2x^2 + 1 = 3x^2 + 4$, and we **can not add** the terms $3x^2$ and 4

Exercise 6

Collect like terms to simplify each expression:

a. $x^2 + 3x - x + 3x^2 =$ **3x – 4 – (x + 1) =**

b. $5y + 3x + 2y + 4x =$

c. $(2x + 3) - (5x - 7) - (x - 1) =$

d. $\frac{2}{3}x^2 + \frac{1}{2}x - \frac{3}{2}x^2 - \frac{1}{5}x + 2 =$

e. $(x + 2) + (2x + 7) - (3x + 4)$

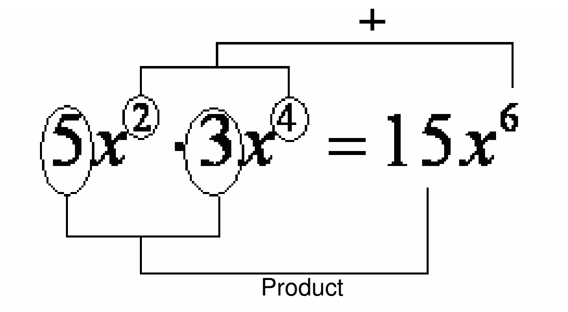
f. $(x^2 + 2x) - (2x^2 - x) + (3x^2 + 5x) =$

g. $(x^2 + y) + (7x^2 - 3y) - (x^2 + 7y) =$

h. $[(3x^2 + 2x^2) - (5x^2 - x)] + \left(8x^2 + \frac{5}{2}x\right) =$

5. Product of monomials.

If you want to multiply two or more monomials, you just have to multiply the coefficients, and add the exponents of the equal letters:



Look at these examples:

a. $(2xy^2)\cdot(-5x^2y) = -10x^3y^3$

b. $3a^2 \cdot 2ab = 6a^3b$

Exercise 7 Multiply

a) $(5x)\cdot(3x)$

b) $(2x)\cdot(3x^2)$

c) $(-2x)\cdot(x^3)$

d) $(4x\ y)\cdot(2x^2y)$

e) $(-7x^2y)\cdot(xy^2)$

f) $(5x^3y^2)\cdot(xy)$

g) $\left(\frac{2}{3}x^3y\right)\cdot(3x)$

h) $(3x)\cdot(2x^2)\cdot(5x^3)$

i) $(x^2)\cdot(-2x)(3x^3)$

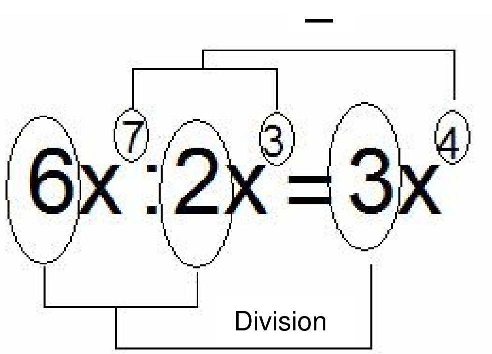
j) $(3ab)\cdot(2a^2)$

k) $(2a^2b^3)\cdot a$

l) $(2x^2z)\cdot(3zx)$

6. Quotient of monomials.

If you want to divide two monomials, you just have to divide the coefficients, and subtract the exponents of the equal letters. The quotient of monomials gives an expression that is not always a monomial.



You can also simplify the fractions that result from the division. Look at these examples:

a. $(10x^3):(2x)=5x^2$

b. $8x^2y:6y^3=\dfrac{8x^2y}{6y^3}=\dfrac{2\cdot 4x\cdot x\cdot y}{2\cdot 3\cdot y\cdot y\cdot y}=\dfrac{4x^2}{3y^2}$ This expression is not a monomial

Exercise 8 Operate

a) $\left(15x^3\right):\left(3x^2\right)$

b) $\left(2x^4\right):\left(3x\right)$

c) $\left(6x\right):\left(2x^3\right)$

d) $\left(12a^2b\right):3a$

e) $15a^3b^2:ab$

f) $3x^5y^2:x^2y^2$

g) $\left(5xy\right)\cdot\left(2x^2y\right):3x$

h) $\left(xy^3\right)\cdot\left(3x^2y\right):2x^2y^2$

i) $\left[\left(3x^4\right):\left(2x\right)\right]\cdot\left(4x^2\right)$

j) $\left(3ab\right):\left(2a^2\right)$

k) $\left(2a^4b^2\right)\cdot 7a$

l) $\left(12x^5yz\right):\left(3xz^3\right)$

7. Polynomials.

A polynomial is the addition or subtraction of two or more monomials.

- If there are two monomials, it is called a **binomial**, for example $x^2 + x$
- If there are three monomials, it is called a **trinomial**, for example $2x^2 - 3x + 1$

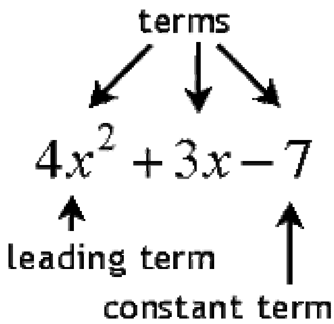
The following are NOT polynomials:

$\frac{1}{x}$ $\sqrt{x^3 - 4}$ $x^2 + 3x + 2x^{-2}$

The **degree** of the entire polynomial is the degree of the highest-degree term that it contains, so

$x^2 + 2x - 7$ is a second-degree trinomial, and $x^4 - 7x^3$ is a fourth-degree binomial.

The polynomial that follows is a second-degree polynomial, and there are three terms: $4x^2$ is the **leading term**, and (-7) is the **constant term**.



Polynomials are usually written this way, with the terms written in "**decreasing**" order; that is, with the highest exponent first, the next highest next, and so forth, until you get down to the constant term.

Polynomials are also sometimes named for their degree:

- a second-degree polynomial, such as $4x^2$, $x^2 - 9$, or $ax^2 + bx + c$, is also called a **"quadratic"**
- a third-degree polynomial, such as $-6x^3$ or $x^3 - 27$, is also called a **"cubic"**
- a fourth-degree polynomial, such as x^4 or $2x^4 - 3x^2 + 9$, is sometimes called a **"quartic"**

Exercise 9 Name the variables, degree, principal term, constant term of the following polynomials

a) $5x^3 + 2x$

b) $x^2 - 4$

c) $3x^5 - 2x^2 - 3x - 1$

d) $3+2x+\frac{7}{3}x^3$

e) 5

f) $-\frac{2}{3}\sqrt[3]{x^2}+x+2$

g) $\sqrt[7]{22}x^5y^8+2x^3y+2xy-y-3$

h) $0x^5-2x^3-4x-x$

8. Evaluating polynomials.

"Evaluating" a polynomial is the same as calculating its number value at a given value of the variable: you substitute the given value of *x*, and calculate the value of the polynomial. For instance:

Evaluate $2x^3 - x^2 - 4x + 2$ at $x = -3$

Plug in –3 for *x*, remembering to be careful with brackets and negatives:

$$\begin{aligned} &2(-3)^3 - (-3)^2 - 4(-3) + 2 \\ &= 2(-27) - (9) + 12 + 2 \\ &= -54 - 9 + 14 \\ &= -63 + 14 \\ &= \mathbf{-49} \end{aligned}$$

Always remember to be careful with the minus signs!

Exercise 10 Evaluate the polynomials:

x^3-2x^2+3x-4 , $5x^3+2x$, $3x^5-2x^2-3x-1$ **and** $3+2x+\frac{7}{3}x^3$ **at the given values of x:**

a) x = 0

b) x = 1

c) x = 2

d) x = -1

9. Adding polynomials.

When adding polynomials you must add each like term of the polynomials, that is, monomials that have the same literal part, you use what you know about the addition of monomials.

There are two ways of doing it. The format you use, **horizontal** or **vertical**, is a matter of preference (unless the instructions explicitly tell you otherwise). Given a choice, you should use whichever format you're more comfortable with.

Note that, for simple additions, **horizontal addition** (so you don't have to rewrite the problem) is probably the simplest, but, once the polynomials get complicated, vertical addition is probably the safest (so you don't "drop", or lose, terms and minus signs). Here is an example:

Simplify $(3x^3 + 3x^2 - 4x + 5) + (x^3 - 2x^2 + x - 4)$

Horizontally:

$$\begin{aligned}&(3x^3 + 3x^2 - 4x + 5) + (x^3 - 2x^2 + x - 4)\\ &= 3x^3 + 3x^2 - 4x + 5 + x^3 - 2x^2 + x - 4\\ &= 3x^3 + x^3 + 3x^2 - 2x^2 - 4x + x + 5 - 4\\ &= \mathbf{4x^3 + x^2 - 3x + 1}\end{aligned}$$

Vertically:

$$\begin{array}{r} 3x^3 + 3x^2 - 4x + 5 \\ x^3 - 2x^2 + x - 4 \\ \hline \mathbf{4x^3 \ + x^2 - 3x + 1} \end{array}$$

Either way, I get the same answer: $\mathbf{4x^3 + x^2 - 3x + 1}$.

Exercise 11

Calculate the following additions of polynomials:

1. $(14x + 5) + (10x + 5)$

2. $(10x + 12) + (6x + 20)$

3. $(19x^2 + 12x + 12) + (7x^2 + 10x + 13)$

4. $(17x^2 + 20x + 11) + (15x^2 + 11x + 17)$

5. $(-15x^2 - 5x + 9) + (-6x^2 - 19x - 16) + (-15x^2 - 14x - 13) + (9x^2 - 14x + 20)$

6. $(-13x^2 - 13x - 10) + (19x^2 - 19x - 5)$

7. $(4x^2 - 6x + 7) + (-19x^2 - 15x - 18)$

8. $(-13x^2 - 5x - 14) + (-14x^2 - 20x + 8)$

9. $(9x^5 - 14x^4 + 18x) + (-6x^5 - 12x^4 - 9x)$

10. $(-20x^2 + 13x - 4) + (11x^2 - 13x - 10)$

10. Subtracting polynomials.

When subtracting polynomials you must realize that a subtraction is the addition of the first term and the opposite of the second:

$$\mathbf{A - B = A + (- B)}$$

Notice that running the negative through the brackets changes the sign on each term inside the brackets. Look at this example:

Simplify $(6x^3 - 2x^2 + 8x) - (4x^3 - 11x + 10)$

Horizontally:

$$\begin{aligned}
&(6x^3 - 2x^2 + 8x) - (4x^3 - 11x + 10) \\
&= (6x^3 - 2x^2 + 8x) - 1(4x^3 - 11x + 10) \\
&= 6x^3 - 2x^2 + 8x - 4x^3 + 11x - 10 \\
&= 6x^3 - 4x^3 - 2x^2 + 8x + 11x - 10 \\
&= \mathbf{2x^3 - 2x^2 + 19x - 10}
\end{aligned}$$

Vertically:

Write out the polynomials, leaving gaps when necessary, and change the signs in the second line. Then add:

$$\begin{array}{rrrr}
6x^3 & - 2x^2 & + 8x & \\
- 4x^3 & & + 11x & - 10 \\
\hline
\mathbf{2x^3} & \mathbf{- 2x^2} & \mathbf{+ 19x} & \mathbf{- 10}
\end{array}$$

Either way, I get the same answer: $\mathbf{2x^3 - 2x^2 + 19x - 10}$

Exercise 12

Calculate the following subtractions:

1. (6x + 14) - (9x + 5)

2. (6x + 19) - (14x + 5)

3. $(14x^2 + 13x + 12) - (7x^2 + 20x + 4)$

4. $(19x^2 + 9x + 16) - (5x^2 + 12x + 7)$

5. $(15x^2 - 9x + 9) - (13x^2 + 15x + 5) - (-16x^2 + 20x + 16) - (-7x^2 + 10x - 10)$

6. $(-9x^2 - 4x - 4) - (-9x^2 - 11x + 12)$

7. From $19x^2 + 11x + 15$ subtract $-5x^2 - 6x - 6$

8. $(-18x^2 + 7x - 14) - (-20x^2 + 17x - 12)$

9. Subtract $-10x^2 + 16x - 15$ from $-6x^2 + 19x + 19$

10. $(17x^5 + 19x^2 + 15) - (9x^7 - 11x - 17)$

11. Multiplying polynomials.

- The first step up in complexity is a monomial times a multi-term polynomial. To do this, I have to distribute the monomial through the brackets. For example:

  $-3x(4x^2 - x + 10) = -3x(4x^2) - 3x(-x) - 3x(10) = \mathbf{-12x^3 + 3x^2 - 30x}$

- The second step in complexity is a multi - term polynomial times a multi - term polynomial. Look at these examples:

  1. Simplify $3(x + 2) = 3\cdot x + 3\cdot 2 = 3x + 6$

  2. Simplify $3x^2\cdot(x + 5) = 3x^2\cdot x + 3x^2\cdot 5 = 3x^3 + 15x^2$

  3. Simplify $(x + 3)(x + 2)$

  The first way I can do this is "horizontally", where I distribute twice:

$$(x+3)(x+2) = x(x) + 3(x) + x(2) + 3(2) = x^2 + 3x + 2x + 6 = \mathbf{x^2 + 5x + 6}$$

The "vertical" method is much simpler, because it is similar to the multiplications learnt at primary school:

4. Simplify (x – 4)(x – 3)

$$\begin{array}{r}
x - 4 \\
x - 3 \\
\hline
x^2 - 4x \qquad\quad \\
-3x + 12 \\
\hline
x^2 - 7x + 12
\end{array}$$

5. Simplify (4x^2 – 4x – 7)(x + 3)

$$\begin{array}{r}
4x^2 - 4x - 7 \\
x + 3 \\
\hline
4x^3 - 4x^2 - 7x \qquad\quad \\
12x^2 - 12x - 21 \\
\hline
4x^3 + 8x^2 - 19x - 21
\end{array}$$

Multiply out the brackets means the same as:

- **Remove** the brackets
- **Expand** the brackets (or the expression)

Exercise 13 Expand

1. $2\ (x + 5)$

2. $5\ (3x - 2)$

3. $4\ (2x - 3)$

4. $-8\ (x + 2)$

5. $-4\ (x - 5)$

6. $9\ (3x + 2)$

7. $-6\ (5x - 4)$

8. $x\ (x +7)$

9. $2x\ (3x + 2)$

10. $-3x^2\ (x + 2)$

11. $-2x\ (5x^2 + 3y^2)$

12. $3x^2y\ (2x + 3y)$

13. $-2x^5\ (7x^2+3)$

    12. $7x^2\ (4x^2 + 2x - 4)$

Exercise 14 Remove the brackets:

1. $(11x + 11)\ (11x + 10)$

2. $(9x + 9)\ (3x + 3)$

3. $(8x + 11)\ (5x + 11)$

4. $(9x + 7)\ (6x + 4)$

5. $(11x + 5)\ (-11x + 12)$

6. $(8x + 11)\ (-3x + 6)$

7. $(-2x^2 - 4x + 11)\ (5x - 12)$

8. $(-11x + 3)\ (-10x^2 - 7x - 9)$

9. $(4x^2 + 12x + 10)\ (-9x^2 + 8x + 2)$

10. $(7x^2 - 6x - 8)\ (-2x + 2)$

11. $(10x^5 + 3)\ (-2x^2 - 11x + 2)$

12. $(-12x - 3)\ (12x^2 - 11x + 3)$

12. Factorising.

Factorising is the reverse process of multiplying out a bracket. The factorised expression has a polynomial inside a bracket, and a term outside.
This term outside must be a **common term** (a number or a letter) . It means that the number or the letter (s) can be found in every term of the expression.
The trick is to see what can be factored out of every term in the expression.
Just don't make the mistake of thinking that "factoring" means "dividing off

and making disappear". Nothing disappears when you factor; things just get rearranged. Here are some examples of how to factor:

- $3x - 12 = \mathbf{3(x - 4)}$
- $12y^2 - 5y = \mathbf{y(12y - 5)}$
- $3x^3 + 6x^2 - 15x = \mathbf{3x(x^2 + 2x - 5)}$

> **Remember**: when the term to be factored out coincides with one of the addends, the unit always remains:
> $$x^2 + x = x(x+1)$$

Exercise 15 Factorise:

1. $12x + 3x$
2. $2xy - 3x$
3. $2x^2 + 18x$
4. $6x^3 - 2x^2$
5. $3x^2y + 2x$
6. $5x+15$

Exercise 16 Factorise:

1. $-336x^3 + 288x$
2. $3x^2 - 3x$
3. $-3x^3 - 33x$
4. $-15x^2 + 18x$

5. $4x^3 - 4x$

6. $160x^3 + 100x^2 - 180x$

7. $19x^3 - 19x$

8. $-6x^3 + 8x$

9. $36x^3 - 24x^2 + 8x$

10. $-14x^2 + 16x$

13. Three algebraic identities.

There are three formulas about operations with binomials that are very common and it is useful to memorise.

1. Square of an addition.

We want to find a formula to work out $(a+b)^2$ where a and b can be numbers or any kind of monomial.

$(a+b)^2 = (a+b)(a+b)$ Let's do it carefully

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

So $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$

Note that $(a+b)^2$ is not the same as $a^2 + b^2$ so don't forget the term $2ab$

2. Square of a subtraction.

Now we want a formula to expand $(a-b)^2$

$$\begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \\ - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array}$$

$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$

Note again that $(a-b)^2$ is not the same as a^2-b^2 but something absolutely different

3. Difference of two squares

What happens if we multiply $(a+b)(a-b)$?

$$\begin{array}{lll} a+b & & \\ a-b & & \\ \hline a^2 & -ab & \\ & +ab & -b^2 \\ \hline a^2 & & -b^2 \end{array}$$

The terms ab and – ab cancel out, so $(a+b)(a-b)=a^2-b^2$

We can use these three formulas in two directions to expand or to factorise any algebraic expression that fits with any of them.

Exercise 17 Calculate using a formula.

1. $(x+2)^2$

2. $(x+4)^2$

3. $(2x+1)^2$

4. $(x-3)^2$

5. $(3x+1)^2$

6. $(2x-5)^2$

7. $(a+2b)^2$

8. $(c^2-2a)^2$

9. $(x-2)(x+2)$

10. $(2x+1)(2x-1)$

11. $(3+x)(3-x)$

12. $(3y-2x)(3y+2x)$

Exercise 18 Factorise:

1. x^2+6x+9

2. x^2+2x+1

3. x^2-2x+1

4. x^2-9

5. $4x^2+4x+1$

6. $x^2+10x+25$

7. x^2-25

8. $4x^2-9$

9. $1-x^2$

10. $2x^4+4\sqrt{2}x^2+4$

7 Equations

Keywords			
Equation	**linear equation**	**quadratic equation**	**term**
Variable	**unknown**		

1 Definition

An equation is a statement that says that two algebraic expressions are equal.

In an equation the equality is true only for particular values of the variables.

In an identity the equality holds for all the values of the variables.

Example 1 the equality $3x - 7 = 2$ is only true when $x = 3$ this is the solution of the equation.

If the degree of the polynomials is 1 the equations are called **linear** equations or first degree equations.

If the degree of the polynomials is 2 the equations are called **quadratic** equations or second degree equations.

Example 2

$5x - 7 = 2x + 3$ Is a linear equation

$5x^2 - 7 = 5(2x + 3)$ Is a quadratic equation

2 Linear equations language in equations

Let's work with an example

$$5x - 3 = 3x + 2$$

- In an equation there is always an equal sign.
- There are two sides with an algebraic expression on each side, left side and right side.
- Each monomial is a term, so $5x$, -3, $3x$ and 2 are **terms** of this equation. In linear equations we will have x-terms and numerical terms.

- x is the variable or unknown.

3 Solving easy equations, basic rules

These rules must be followed, usually in this order, to solve linear equations.

Rule 1 If there are brackets we must remove them (expand the expression).

Rule 2 Any x-term or number that is adding (positive sign) moves to the other side subtracting (negative sign) and vice versa.

Rule 3 If there are x-terms on both sides, collect them on one side and do the same with the numbers.

Rule 4 A number that is multiplying (the whole expression) on one side moves to the other side dividing and vice versa.

Examples 3

a) Solving $3x - 2 = 12$, there are no brackets.

$3x - 2 = 12 \Rightarrow 3x = 12 + 2$ (Rule 2)

$3x = 14$ (Rule 3)

$x = \frac{14}{3}$ (Rule 4), this is the solution.

b) $5x - 3 = 2x + 1$, there are no brackets.

$5x - 2x = 1 + 3$ (Rule 2)

$3x = 4$ (Rule 3)

$x = \frac{4}{3}$ (Rule 4), this is the solution.

c) $13x + 2 = 5(2x + 1)$

$13x + 2 = 10x + 5$ (Rule 1)

$13x - 10x = 5 - 2$ (Rule 2)

$3x = 3$ (Rule 3)

$x = \frac{3}{3} \Rightarrow x = 1$ (Rule 4), this is the solution.

d) $\frac{x}{3} - 1 = 4$; $\frac{x}{3} = 4 + 1$; $\frac{x}{3} = 5$; $x = 5 \cdot 3$; $x = 15$ (sol.)

Of course you don't need to write the rule you are using during the process.

Exercises 1

Solve:

1) $x + 12 = 3$

2) $x + 5 = 7$

3) $5 = x + 2$

4) $3 = 2 - x$

5) $2x = 10$

6) $5x = 4$

7) $\frac{x}{2} = 10$

8) $-3 = 2x$

9) $3x + 6 = 12$

10) $1 = 2x - 4$

11) $2 = 3x - 1$

12) $3 = 2 - 4x$

13) $2x + 1 = x + 2$

14) $3x + 2 = 5x - 3$

15) $2x - 3 = x$

16) $6x - 5 = 3x - 1$

17) $6x + 2 = 2 - 3x$

18) $-2x+1=-3x+2$

19) $5x-3=2-4x$

20) $6x-4=15x-3$

21) $3x-2+4x=5x-2$

22) $x+1-3=2x+4-x$

23) $5x-3=2(x+1)$

24) $6(x-2)=7x-3$

25) $2(5x-3)=x+1$

26) $3(5x-1)=2(2-3x)$

27) $3(x+2)+x=2x-3$

28) $\dfrac{x}{4}+2=1$

29) $\dfrac{2x}{5}=3$

30) $\dfrac{3x}{2}+5=-2$

31) $\dfrac{5x}{3}=-2$

32) $8\left(\frac{x}{4}-2\right)=\frac{3}{2}$

33) $2\left(\frac{x}{2}-\frac{2}{3}\right)=\frac{3}{5}$

4 Equations with denominators

When there are denominators in several terms of the equation, the first thing we need to do is to simplify multiplying every term by the lowest common multiple of the denominators, if there is no denominator in a term you should consider it as 1; then we continue with the same rules that we have seen before.

Example 4 solving $x-\frac{2}{3}=\frac{5x}{2}-1$

The L.C.M. of 3 and 2 is 6, so we multiply both sides by 6

$6\left(x-\frac{2}{3}\right)=6\left(\frac{5x}{2}-1\right)$; $6x-4=15x-6$, and then:

$$6x-15x=-6+4 \Rightarrow -9x=-2 \Rightarrow x=\frac{-2}{-9} \Rightarrow \boxed{x=\frac{2}{9}}$$

Solving a complex equation requires organising the calculations in these steps

1. If there are brackets remove them (expand)
2. If there are denominators remove them (multiplying both sides by the LCM of the denominators)
3. Transpose the like terms (move to one side the x-terms and the numbers to the other side)

4. Combine like terms
5. Isolate the unknown (move the coefficient of the x to the other side)

Exercises 2

Solve:

1) $-\frac{3}{5}+\frac{x}{10}=-\frac{1}{5}-\frac{x}{5}$

2) $2(3x-1)=3(x-1)$

3) $\frac{x+1}{3}=\frac{x-1}{4}$

4) $4(x-1)-2=3x$

5) $\frac{2x-1}{3}=\frac{x}{2}$

6) $4(1-2x)=3(2-x)$

7) $3(2x+1)+2(x-1)=23$

8) $5(1-2x)-3(4+4x)=0$

9) $\frac{12}{2x-3}=4$

10) $\frac{5}{x+5}=\frac{15}{x+7}$ **11)** $\frac{x}{3}+\frac{x}{4}=1$

Exercises 3 Find the number in each question by forming an equation and solving it.

1) **If you multiply the number by 2 and then subtract 5, the answer is 4**

2) **If you multiply the number by 10 and then add 19, the answer is 16**

3) **If you add 3 to the number and then multiply the result by 4, the answer is 10**

4) **If you subtract 11 from the number and then treble the result, the answer is 20**

5) **If you double the number, add 4 and then multiply the result by 3, the answer is 13**

6) **If you treble the number, take away 6 and then multiply the result by 2, the answer is 18**

7) If you double the number and subtract 7 you get the same answer as when you add 5 to the number.

8) If you multiply the number by 5 and subtract 4, you get the same answer as when you add 3 to the number and then double the result.

9) If you multiply the number by 6 and add 1, you get the same answer as when you add 5 to the number and then treble the result.

10) If you add 5 to the number and then multiply the result by 4, you get the same answer as when you add 1 to the number and then multiply the result by 2.

5 Solving problems using linear equations

So far you have concentrated on solving given equations. Making up your own equations helps you to solve difficult problems.
There are four steps.

I Call the unknown number x (or any other suitable letter) and state the units where appropriate. Write down what we call “x” is.

II Write the problem in the form of an equation. Read the problem carefully

III Solve the equation and give the answer in words.

IV Check your solution using the problem and **not** your equation.

Exercises 4 Solve each problem by forming an equation. The first questions are easy but should still be solved using an equation, in order to practise the method.

1 The length of a rectangle is twice the width. If the perimeter is 20 cm, find the width.

x

2 The width of a rectangle is one-third of the length. If the perimeter is 96 cm, find the width.

3 The sum of three consecutive numbers is 276. Find the numbers.

Let the first number be x.

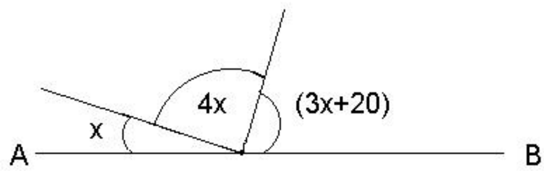
4 The sum of four consecutive numbers is 90. Find the numbers.

5 The sum of three consecutive odd numbers is 177. Find the numbers.

6 Find three consecutive even numbers which add up to 1524.

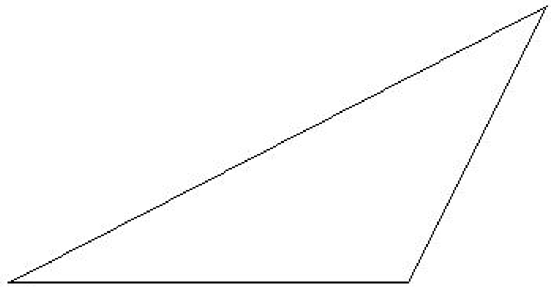
7 When a number is doubled and then added to 13, the result is 38. Find the number.

8 If AB is a straight line, find x.



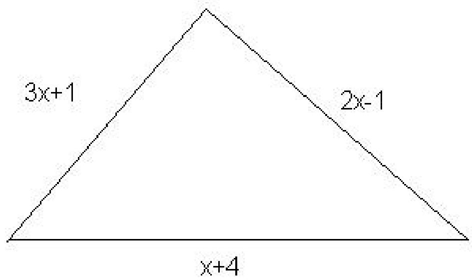
9 The difference between two numbers is 9. Find the numbers, if their sum is 46.

10 The three angles in a triangle are in the ratio 1:3:5. Find the angles.



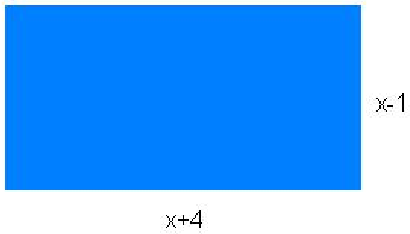
11 The sum of three numbers is 28. The second number is three times the first and the third is 7 less than the second. What are the numbers?

12 If the perimeter of this triangle is 22 cm, , find the length of the shortest side.



13 David weighs 5 kg less than John, who in turn is 8 kg lighter than Paul. If their total weight is 197 kg, how heavy is each person?

14 The perimeter of the rectangle is 34 cm. Find x.

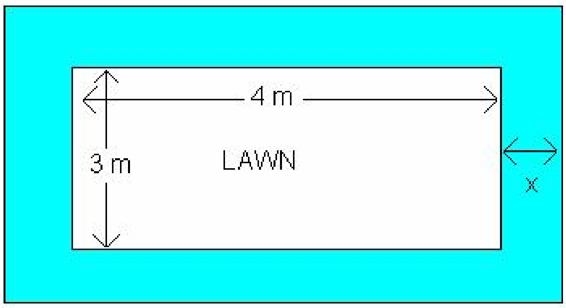


15 The diagram shows a rectangular lawn surrounded by a footpath x m wide.

a) Show that the area of the path is $4x^2 + 14x$

b) Find an expression, in terms of x, for the distance around the outside edge of the path.

c) Find the value of x when this perimeter is 20 m.



16 A man is 32 years older than his son. Ten years ago he was three times as old as his son was then. Find the present age of each one.

17 A man runs to a telephone and back in 900 seconds. His speed on the way to the telephone is 5 m/s and his speed on the way back is 4 m/s. Find the distance to the telephone.

18 A car completes a journey in 10 minutes. For the first half of the distance the speed was 60 km/h and for the second half the speed was 40 km/h. How far is the journey?

19 A bus is travelling with 48 passengers. When it arrives at a stop, x passengers get off and 3 get on. At the next stop half the passengers get off and 7 get on. There are now 22 passengers. Find x.

20 A bus is travelling with 52 passengers. When it arrives at a stop, y passengers get off and 4 get on. At the next stop one-third of the passengers get off and 3 get on. There are now 25 passengers. Find y.

21 Mr Lee left his fortune to his 3 sons, 4 nieces and his wife. Each son received twice as much as each niece and his wife received £6000, which was a quarter of the money. How much did each son receive?

6. Quadratic equations

Quadratic equations always have a x^2 term; in general they have also a x-term and a number. Generally they have two different solutions.

First we are going to solve quadratic equations with only two terms.

6.1 Equations in the form $ax^2 + c = 0$

Examples 5

a) Solve $2x^2 - 8 = 0$; we transpose 8 to the right side

$2x^2 = 8$; We isolate x^2

$x^2 = \frac{8}{2} \rightarrow x^2 = 4$ From here we can take the square root and the solutions are $x = 2$ and $x = -2$

b) Solve $3x^2 + 5 = 0$; we transpose 5 to the right side

$3x^2 = -5 \rightarrow x^2 = -\frac{5}{3}$, there is not any solution because the square of all number gives a positive number.

6.2 Equations in the form $ax^2 + bx = 0$

Examples 6

a) Solve $3x^2 + 6x = 0$

We factorise $x(3x+6) = 0$

The product of two numbers can only be zero if one or both are zero, so

$x(3x+6) = 0 \Rightarrow \begin{cases} x = 0 \\ 3x + 6 = 0 \rightarrow x = -2 \end{cases}$ The solutions are $x = 0$ and $x = -2$

b) Solve $2x^2 - 3x = 0$; $x(2x-3) = 0$; $x = 0$ and $2x - 3 = 0 \rightarrow x = \frac{3}{2}$

Note that in these cases one of the solutions is always $x = 0$.

Exercises 5

Solve:

1) $x^2 - 3x = 0$

2) $2x^2 - 6 = 0$

3) $3x^2 - 5x = 0$

4) $2x^2 + 7 = 0$

5) $x^2 - 9 = 0$

6) $2x^2 - 16x = 0$

6.3 Equations in the form $ax^2 + bx + c = 0$ (complete form)

If we fail to factorise using any simple method we use this formula that has to be memorised.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 7

Solve $x^2 - x - 2 = 0$

Start writing down the values of a, b and c

$a = 1$

$b = -1$ and using the formula $x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1} = \dfrac{1 \pm \sqrt{9}}{2}$

$c = -2$

The solutions are $x_1 = \dfrac{1+3}{2} = 2$, $x_2 = \dfrac{1-3}{2} = -1$

Exercises 6

Solve

1) $x^2 - 3x + 2 = 0$

2) $x^2 - 6x + 5 = 0$

3) $x^2 - 3x - 4 = 0$

4) $2x^2 + 2x - 12 = 0$

5) $2x^2 - x - 6 = 0$

6) $6x^2 + 5x - 6 = 0$

7) $3y^2 - 2y + 1 = 0$

8) $2y^2 - 7y + 2 = 0$

9) $x^2 - 5x + 3 = 0$

10) $(2x - 3)(x + 1) = 2x + 5$

11) $a^2 - 2a(a + 1) = -8$

12) $z^2 - 3z + 2 = 0$

13) $\frac{3z-1}{3} = \frac{2}{z+2}$

14) $\frac{x-1}{2} + \frac{x(x-2)}{3} = x^2 - 1$

15) $x(x-2) - 3(x-1) = x^2 - 2$

Exercises 7

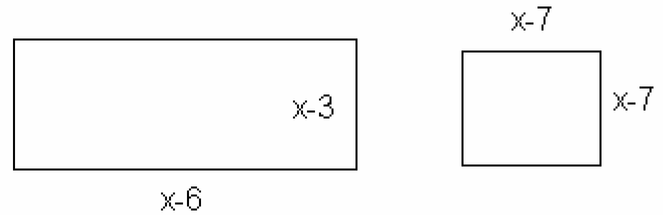
1 Two numbers which differ 3, have a product of 88. Find them

2 The product of two consecutive odd numbers is 143. Find the numbers

3 The height of a photo exceeds the width by 7 cm. If the area is 60 cm^2, find the height of the photo.

4 The length of a rectangle exceeds the width by 2 cm and the diagonal is 10 cm long, find the width of the rectangle.

5 The area of the rectangle exceeds the area of the square by 24m^2. Find x.

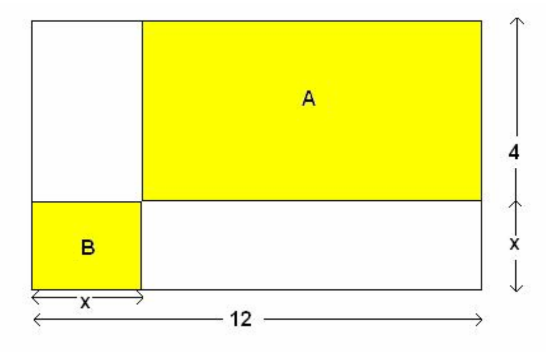


6 Three consecutive integers are written as x, x + 1, x + 2. The square of the largest number is 45 less than the sum of the squares of the other numbers, Find the three numbers

7 (x- 1), x and (x + 1) represent three positive integers. The product of the three numbers is five times their sum.
a) Write an equation in x,
b) Show that your equation simplifies to x^3 -16x = 0.
c) Factorise x^3 -16x completely,
d) Finally find the three positive integers.

8 An aircraft flies a certain distance on a bearing of 45º, and then twice the distance on a bearing of 135°, its distance from the starting point is then 350 km. Find the length of the first part of the journey.

9 The area of rectangle A is twice the area of B. Find x.

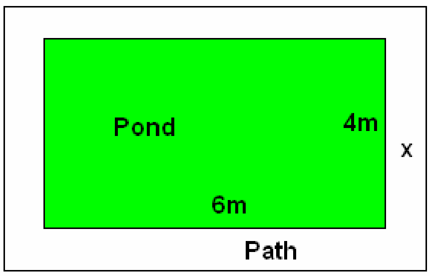


10 The perimeter of a rectangle is 68 cm, if the diagonal is 26 cm, find the dimensions of the rectangle.

11 A stone is thrown in the air. After t seconds its height, h, above sea level is given by the formula $h = 80 + 3t - 5t^2$, Find the value of t when the stone falls into the sea.

12 The total surface area of a cylinder, A, is given by the formula $A = 2\pi r^2 + 2\pi rh$. Given that A = 200cm^2 and h = 10cm, find the value of r, correct to 1 decimal place.

13 A rectangular pond, 6 m x 4 m, is surrounded by a uniform path of width x. The area of the path is equal to the area of the pond. Find x.



14 The perimeters of a square and a rectangle are equal. The length of the rectangle is 11 cm and the area of the square is 4cm2 larger than the area of the rectangle. Find the side of the square.

15 The sequence 3, 8, 15,24, ...can be written (1 x 3), (2 x 4), (3x5),(4x6)...

a) Write an expression for the n^{th} term of the sequence.
One term in the sequence is 255.
b) Form an equation and afterwards find what number term it is.

16 A cyclist travels 40 km at a speed x km/h. Find the time taken in terms of x. Find the time taken when his speed is reduced by 2km/h. If the difference between the times is 1 hour, find the original speed, x km/h.

17 An increase of speed of 4 km/h on a journey of 32 km reduces the time spent by 4 hours. Find the original speed.

18 A train normally travels 60 miles at a certain speed. One day, due to bad weather, the train's speed is reduced by l0 mph, so the journey takes 3 hours longer. Find the normal speed.

19 A number exceeds four times its reciprocal by 3. Find the number.

20 Two numbers differ by 3. The sum of their reciprocals is $\frac{7}{10}$; find the numbers.

24 The numerator of a fraction is 1 less than the denominator. When both numerator and denominator are increased by 2, the fraction is increased by $\frac{1}{12}$ .Find the original fraction.

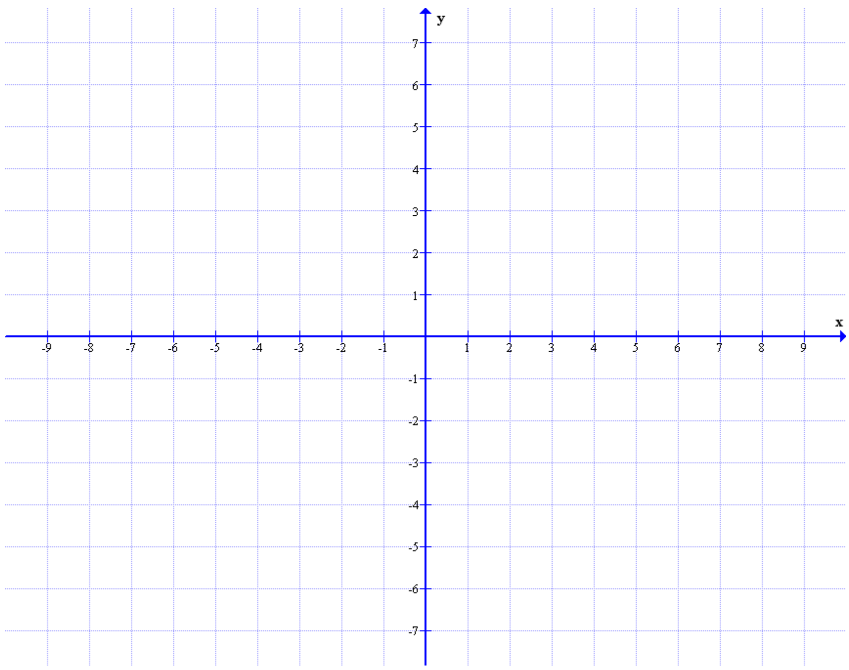
8 Graphs

Keywords			
Coordinate/Cartesian plane/grid		graph	function
x-axis	grid	ordered pair	quadrant
y-axis	origin	points	domain
axes	coordinates	continuous	range
increasing	decreasing	discontinuous	slope
gradient	linear	y-intercept	parallel line

1 Coordinating the plane

1.1 The coordinate grid.
In the sixteenth century, the French mathematician *René Descartes* developed a grid of numbers to describe the location of any point in the plane.

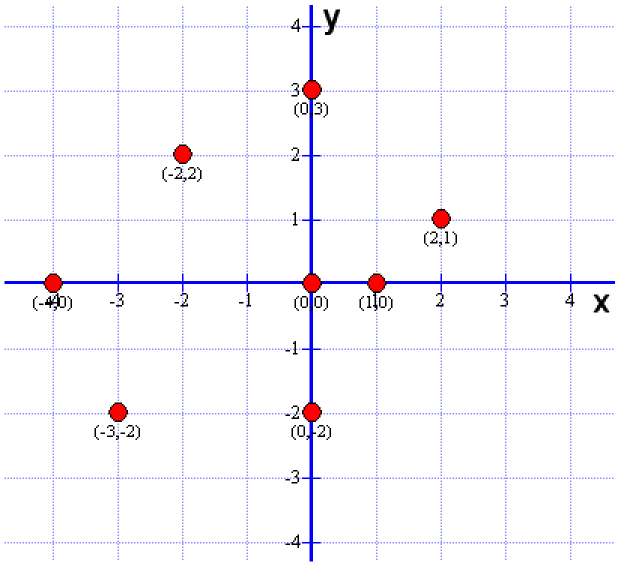
In the diagram below we have a horizontal line and a vertical one meeting at the point 0. It is called the **coordinate plane (or grid)** or the **Cartesian Plane** in honour of Descartes:



The horizontal line is called the **x-axis.**
The vertical line is called the **y-axis.**
The point where the two lines meet is the **origin.**

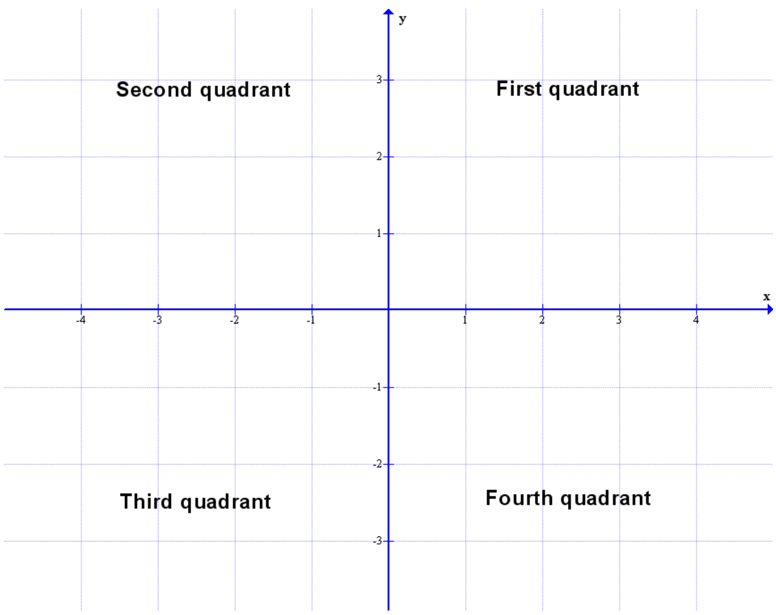
1.2 Plotting points in the plane.

Let’s plot for example the point (2, 1): starting at the origin, we move 2 units to the right along the x-axis; then move vertically up 1 unit. We describe the location of this point as the **ordered pair** (2, 1). This ordered pair is also called as the **coordinates** of the point. When giving the coordinates of a point, the x-value comes first. Some points are plotted in the coordinate plane below:



1.3 The four quadrants

The diagram below shows that the x-axis and the y-axis divide the plane into four quarters. Each of these is called a **quadrant:**



Exercise 1

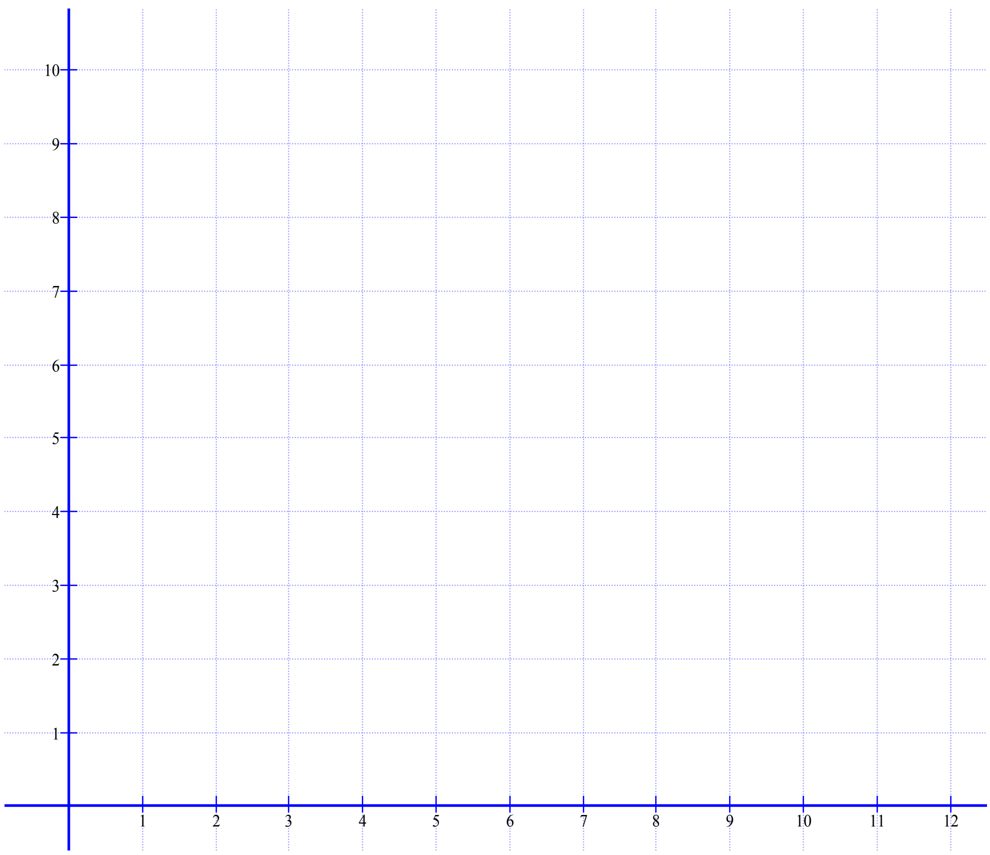
Draw a grid as shown. Join these points in order, using the same grid.

a) (4, 6), (5, 7), (6, 6), (4, 6).

b) (5, 8), (4, 8), (4, 7), (5, 8), (6, 8), (6, 7), (5, 8).

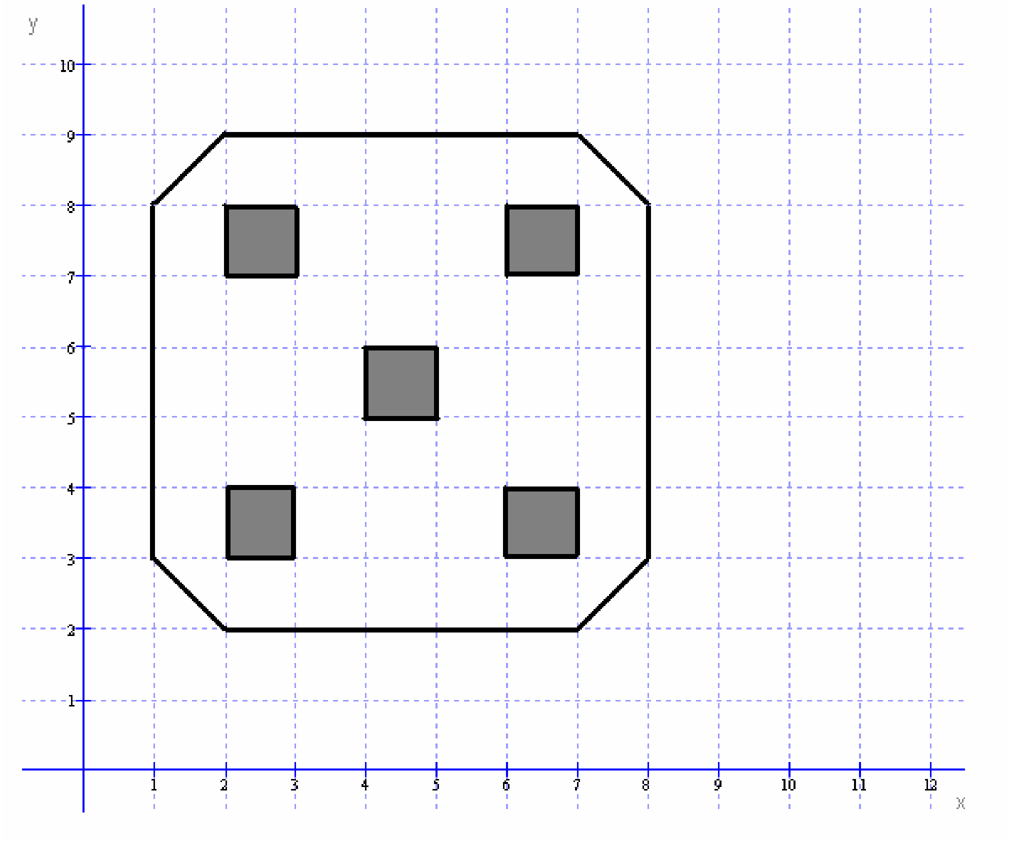
c) (4, 5), (5, 4), (6, 5), (5, 3), (4, 5)

d) (5, 2), (3, 4), (3, 5), (2, 5), (2, 8), (3, 8), (3, 9), (7, 9), (7, 8), (8, 8), (8, 5) (7, 5), (7, 4), (5, 2).



Exercise 2

The diagram below shows the face of a dice showing a 5. Write a set of instructions that would give the face of the dice that shows a 2.

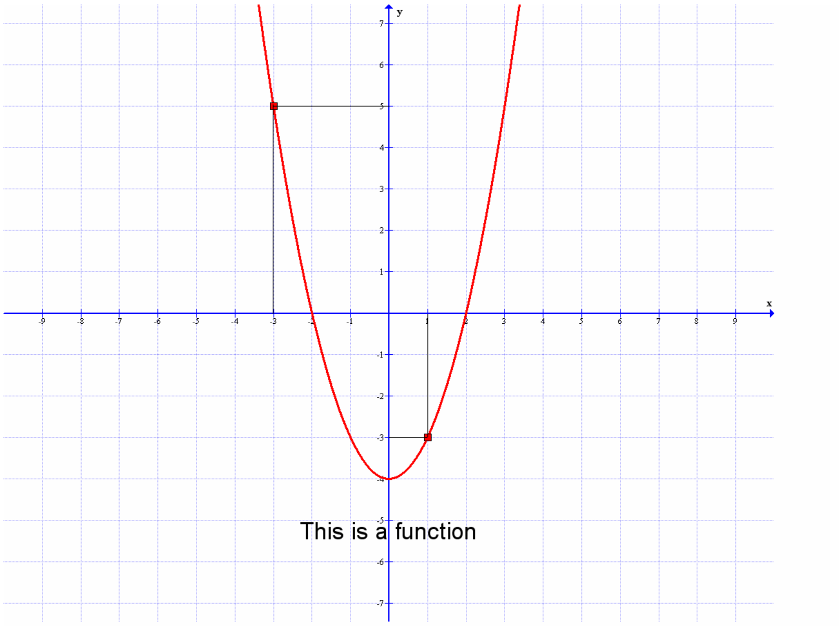


2 Functions

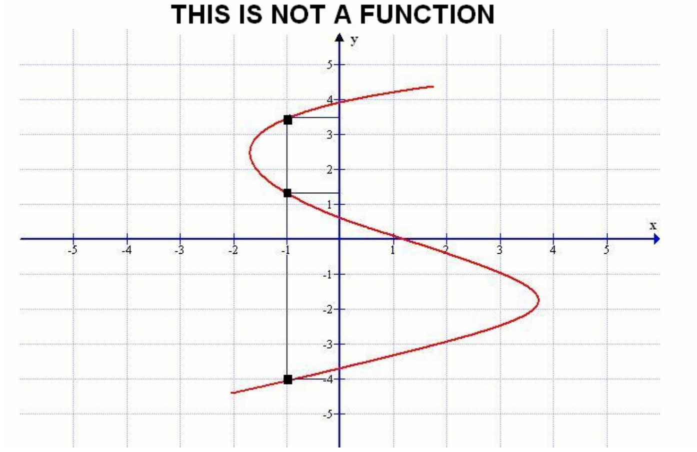
2.1 Definition.

A **function** is a relation between two variables called x and y in which: • x is the independent variable • y is the dependent variable • Every x -value is related to **one and only one** y-value.

Functions can be represented using grids and points. This is important when the function behaviour needs to be visualized. But be careful, because there are graphs which are not functions. For example:

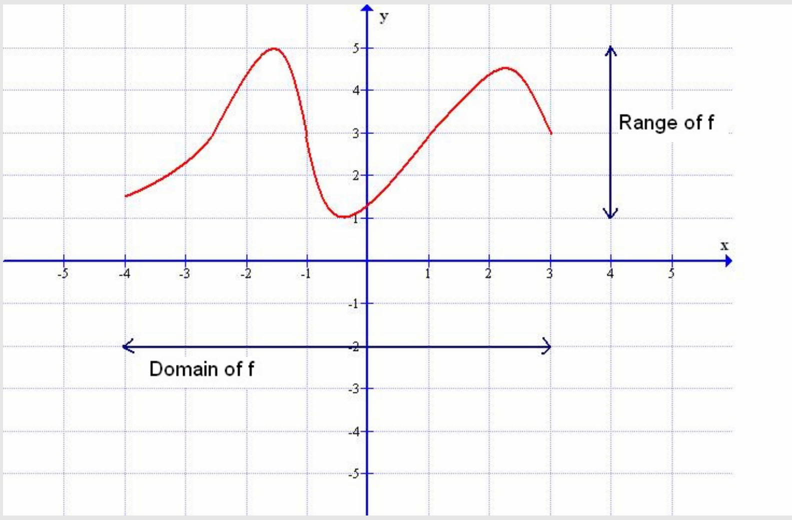


THIS IS NOT A FUNCTION



In the second case, for example the x-value -1 has three different correspondences, so, it is not a function.

The set of all real numbers variable x can take such that the expression defining the function is real is the **domain** of the function, and the set of all values that the function takes when x takes values in the domain is called the **range.** For example:



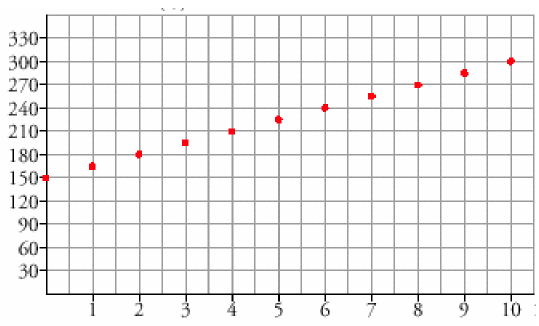
The graph of a function has to be studied from the left to the right, that is to say, how the y-coordinate varies when the x-coordinate increases.

If the graph of a function is going up from left to right, then it is an **increasing** function.
If the graph is going down from left to right, then it is a **decreasing** function.

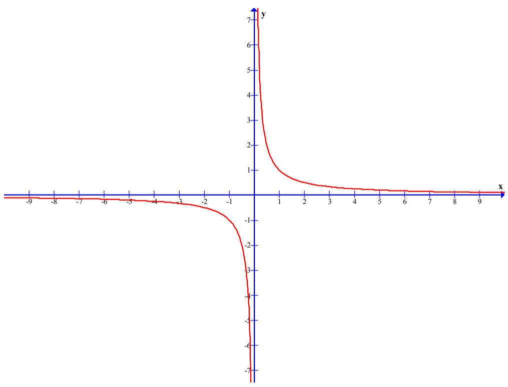
Exercise 3

Find the domains and ranges of the following functions:

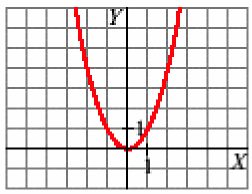
a)



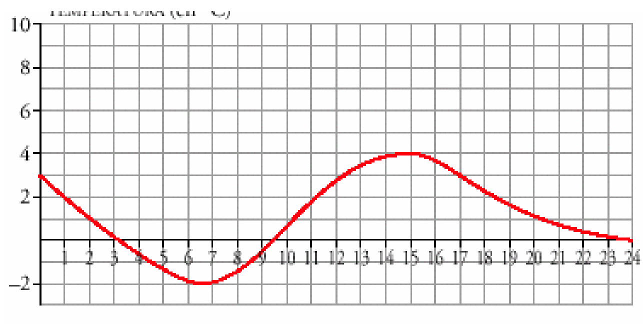
b)



c)

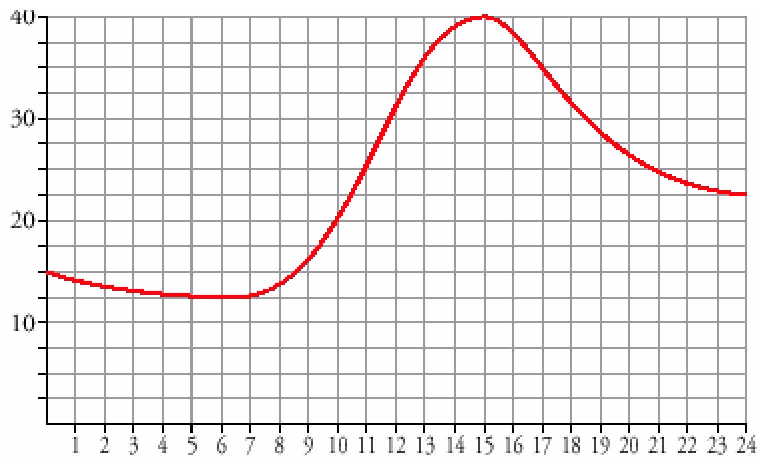


d)



Exercise 4

This function shows the temperature in a city during a day.



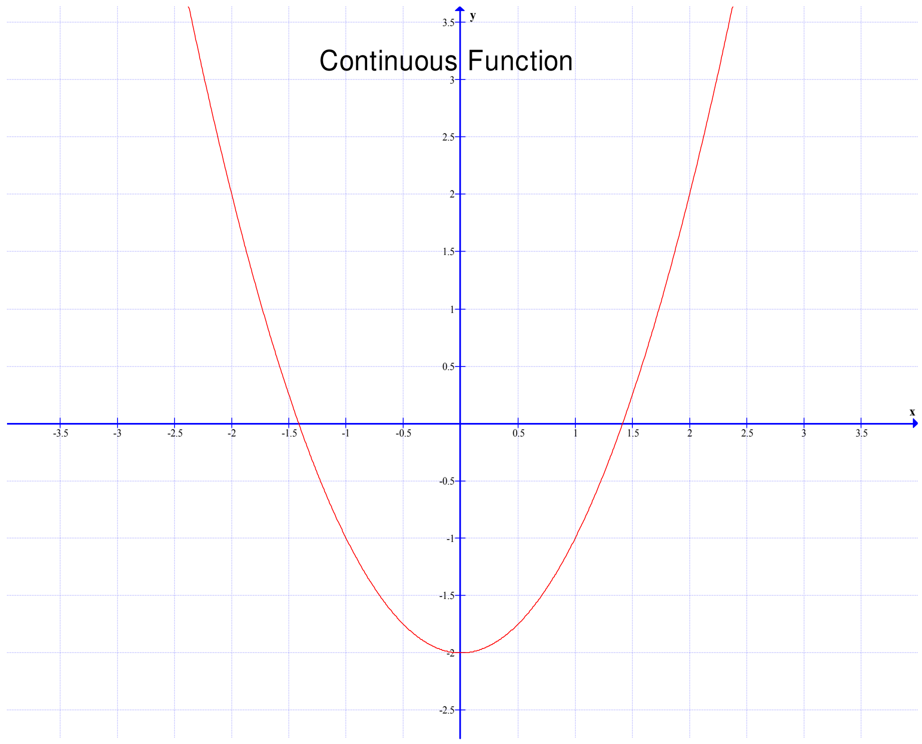
Answer these questions:

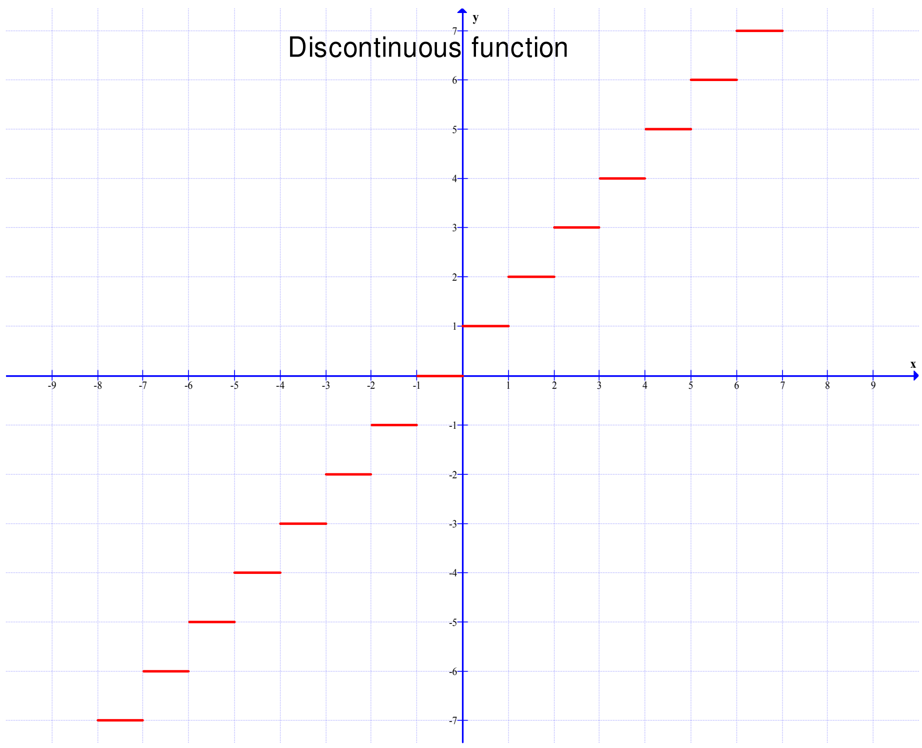
a) **What are the x-variable and the y-variable?**

b) **What is the domain? And the range?**

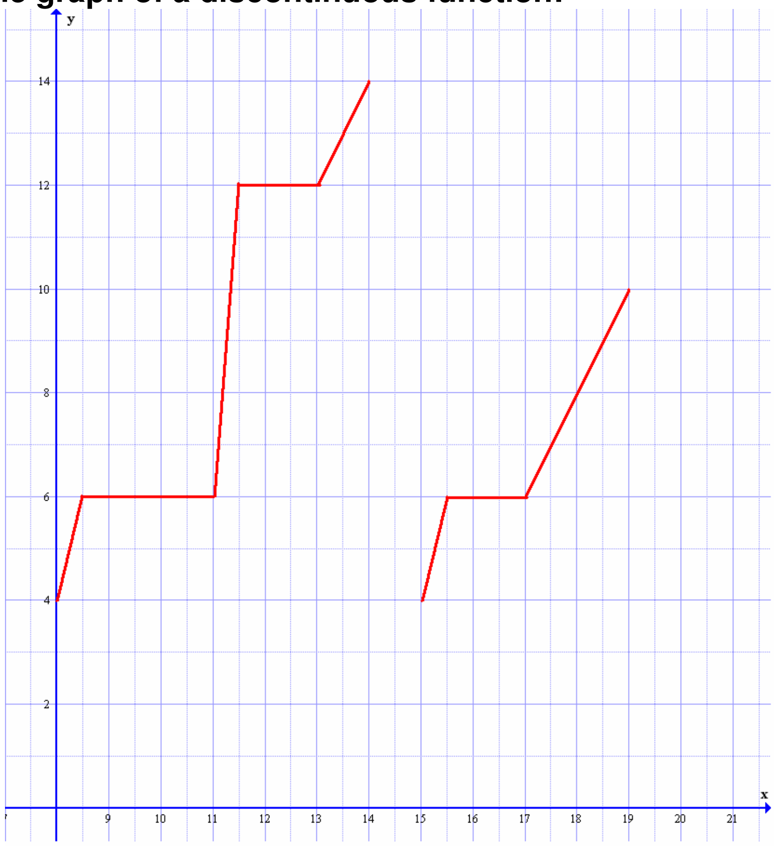
c) **When is the function increasing and decreasing?**

A **continuous function** is a function whose graph can be drawn without lifting the chalk from the blackboard, (or the pen from the notebook). Otherwise, the function is **discontinuous.** This is only an intuitive definition.





Exercise 5
This is the graph of a discontinuous function:



Invent a situation that could be described in the graph.

Exercise 6
This is the price list of a car park:

Open from 9 h. to 22 h.
First and second hours.................................Free
Third hour and consecutive or fraction.........1 € each.
Daily maximum...12 €

Draw the graph of the function which relates park timing to its price. Is it a continuous function?

2.2 Another way of seeing the concept of function.

In mathematics we use the word FUNCTION for any rule, where for any input number; there is only one output number. The input and the output numbers may be written as an ordered pair (or a couple)
A set of couples will be a function if no two couples have the same first component.
Thus, {(1, 4), (2, 5), (3, 6), (4, 7)} is a function, but {(2, 7), (3, 8), (3, 9), (4, 12)} is not a function as (3, 8) and (3,9) have the same first component.
In a set of couples, for example:

{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)}

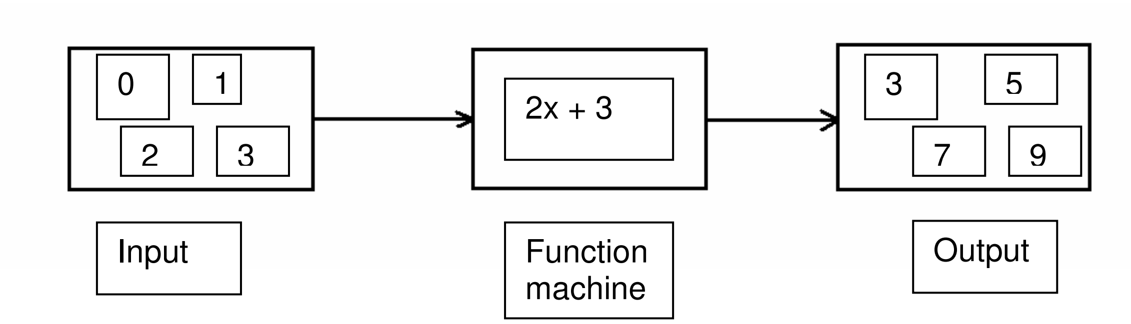
The set of first components is the domain:

Domain: {1, 2, 3, 4, 5}

The set of second components is the range:

Range: {1, 4, 9, 16, 25}

From this, you can see that a function may be thought of as a machine which processes numbers:



2.3 Notation for a function

Notice that all the functions that we have seen until now have been studied with a graph which allows us to know some characteristics of the function. But there is a great amount of functions given by an **analytical expression** which connects x and y variables algebraically.

We generally use the letter *f* to represent a function. If a function tell us to double a number and add 4, i.e. 2x + 4, the function may be written in any of these ways:

i) $f(x) = 2x + 4$
ii) $y = 2x + 4$

Example

Complete this table relating the base and height of rectangles whose area is 12 m²:

Base x (m)	1	2	3	4	6	12	x
Height y (m)							

a. Represent the function in the form of a graph.
b. Find which of these expressions corresponds to this function

$y = \dfrac{x}{12}$ $\qquad$ $y = \dfrac{12}{x}$ $\qquad$ $y = 12x$

Solution:

We complete the table and draw the graph:

Base x (m)	1	2	3	4	6	12	x
Height y (m)	12	6	4	3	2	1	12/x

So, the right expression for this function is $y = \dfrac{12}{x}$

2.4 Drawing the graph of a function given by a formula

To draw the graph of a function:

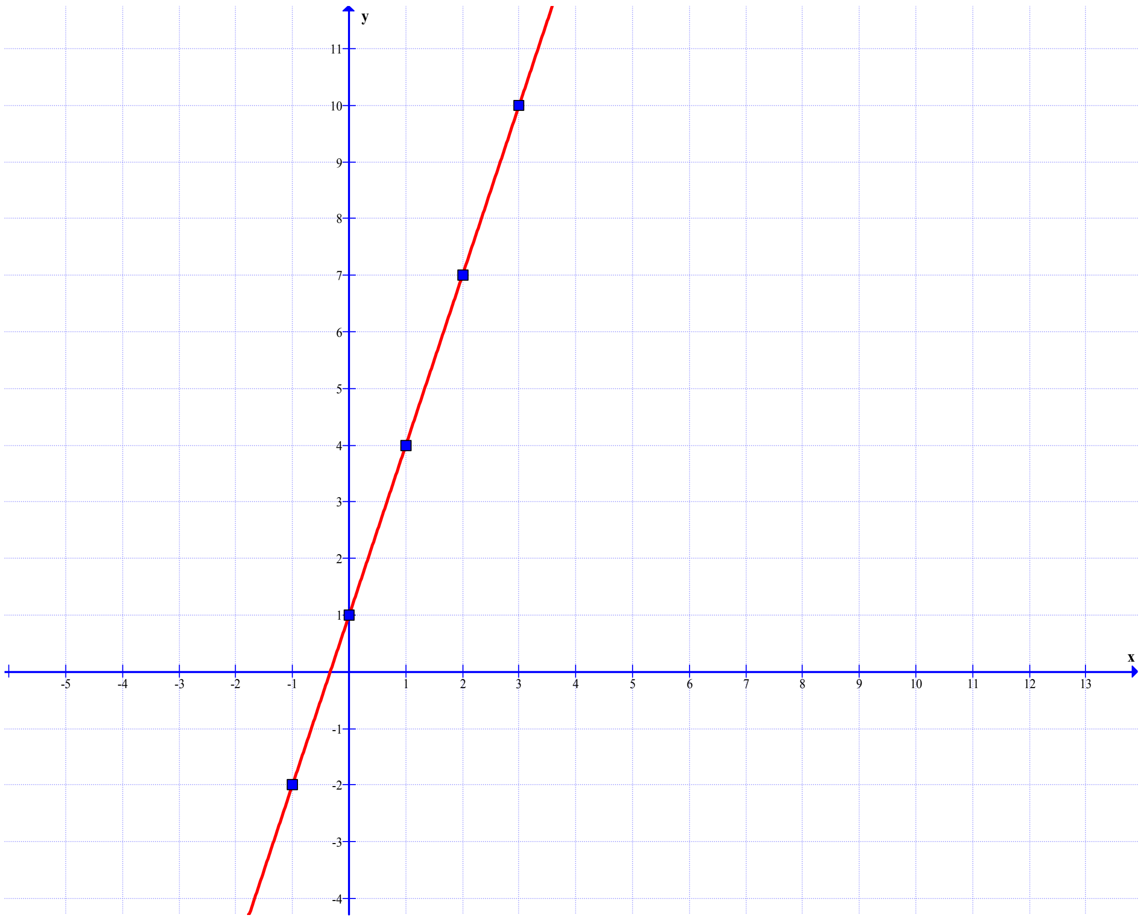
- Write a table of values
- Calculate the value of *y* for each value of *x*
- Draw a suitable grid
- Plot the pairs (x,y) and join them with a line

For example: draw a graph of the function y = 3x +1

First, make a table of easy values for the x-coordinates. Then calculate the corresponding values of y, using y = 3x +1. These y-coordinates are also put in the table:

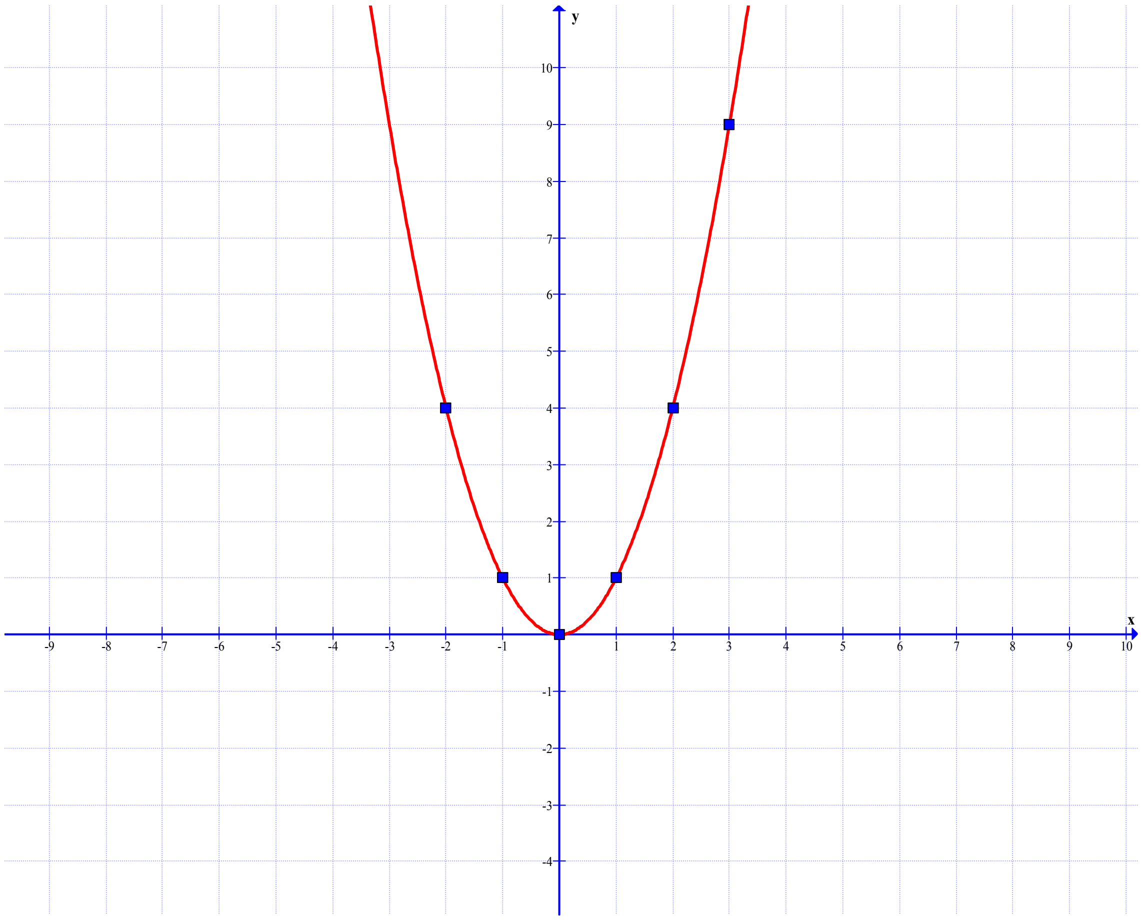
x	-1	0	1	2	3
y	-2	1	4	7	10

Then, construct a grid, plot the coordinates and join them to give a straight line graph.
Note that this graph passes through countless other coordinates, all of which obey the same rule of the function. You can choose any point on the line which has not been plotted to show that this is true.



Example: Make a table of values of the function y = x^2. Then draw the graphs in a suitable grid.

x	-2	-1	0	1	2	3
y	4	1	0	1	4	9



Exercise 7

Copy and complete the table below for the functions:

i) $y = x + 2$
ii) $y = x^2$

x	-2	-1	0	1	2	3
$y = x + 2$						
$y = x^2 - 4$						

Then draw both graphs on a suitable grid

3 Linear graphs

3.1 Functions with the expression y = mx

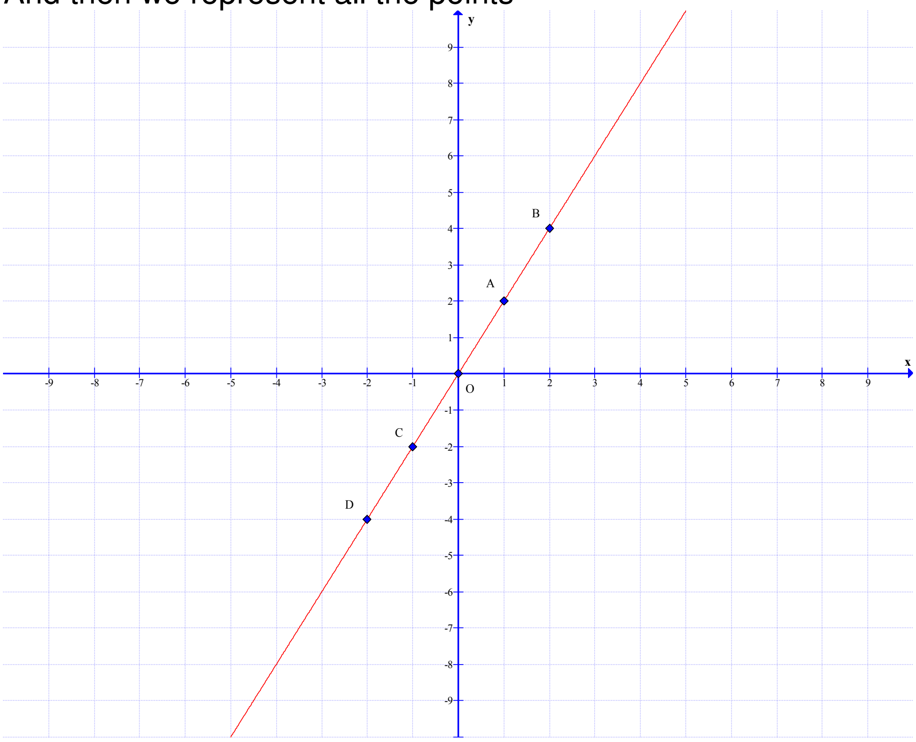
Let's work with some examples.

Draw the graph of $y = 2x$ and on the same grid the graph of $y = 3x$ and compare them.

First we make a table with some points of $y = 2x$

x-value	0	1	2	-1	-2
y-value	0	2	4	-2	-4
Point	O(0,0)	A(1,2)	B(2,4)	C(-1,-2)	D(-2,-4)

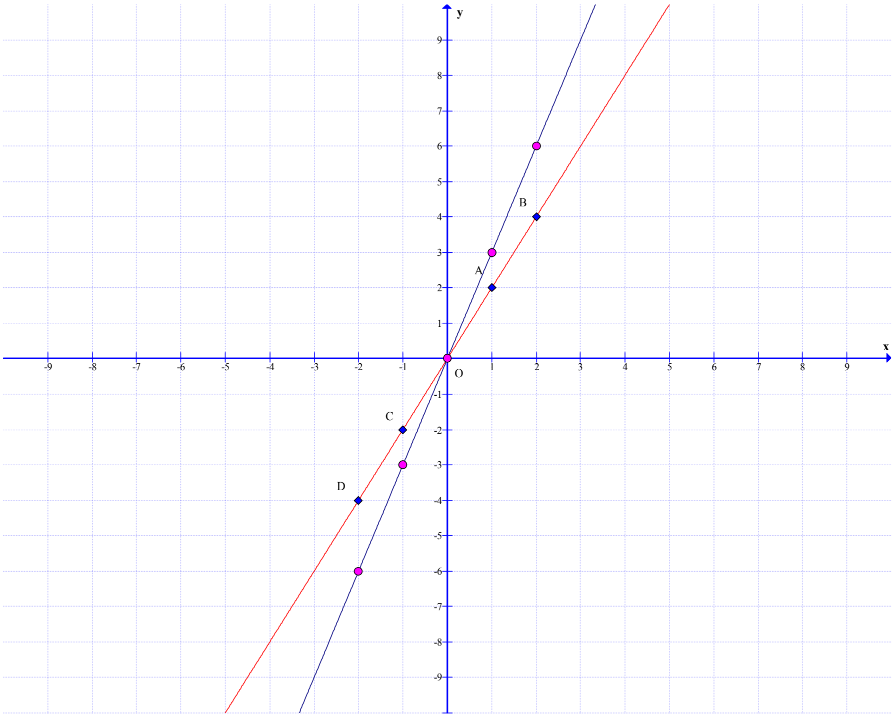
And then we represent all the points



If we make a table with some points of $y = 3x$

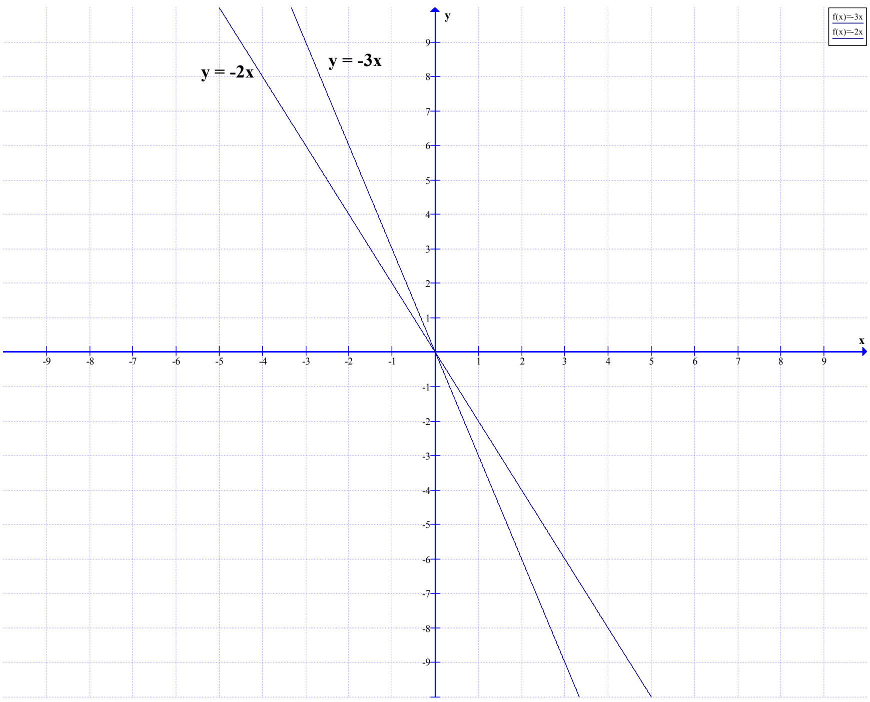
x-value	0	1	2	-1	-2
y-value	0	3	6	-3	-6
Point	O(0,0)	A(1,3)	B(2,6)	C(-1,-3)	D(-2,-6)

And then we represent both graphs



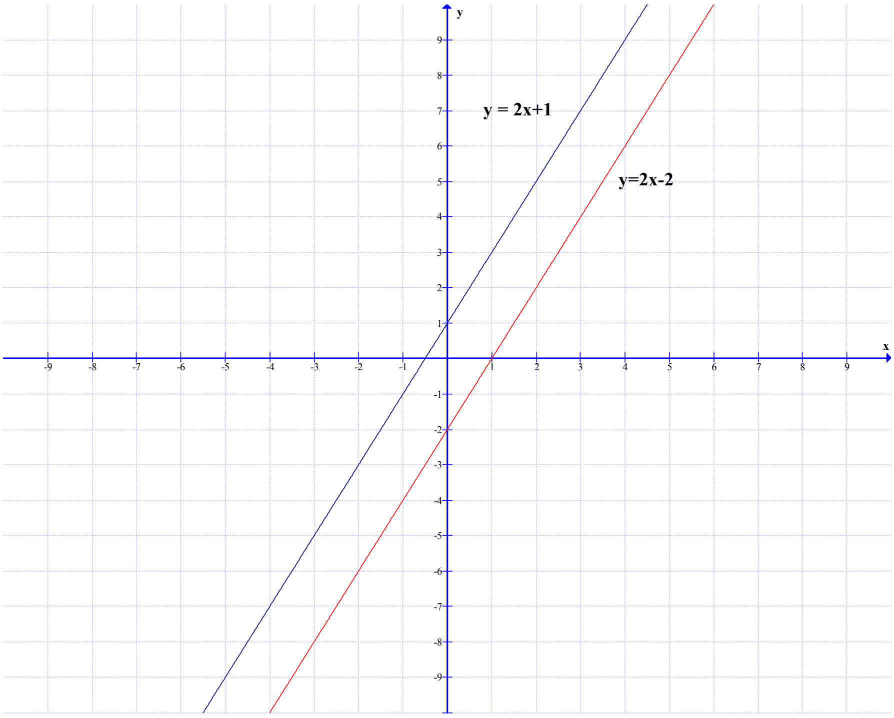
As it can be seen, the line $y = 3x$ is steeper than $y = 2x$, the x-coefficient is the gradient and indicates how steep the line is. All the functions with an expression in the form $y = mx$ go through the origin.

If we represent the lines $y = -3x$ and $y = -2x$, we get:



We get decreasing functions and they go through the origin as well.

If we represent the functions $y = 2x + 1$ and $y = 2x - 2$ we get:



Comparing their graphs we can see that they are straight lines, their gradient is 2, so they are parallel lines, and they cut the y-axis in the points (0,1) and (0,-2) 1 and -2 are called y-intercept.
Summarising all this we make the following definitions:

3.2 Definitions:

All straight lines have a similar expression:

y = mx + n where:

- m is the **gradient** or **slope**, and indicates how steep it is.
- n is the **y-intercept**, the point where the line cuts the y-axis.

Parallel lines have the same gradient.

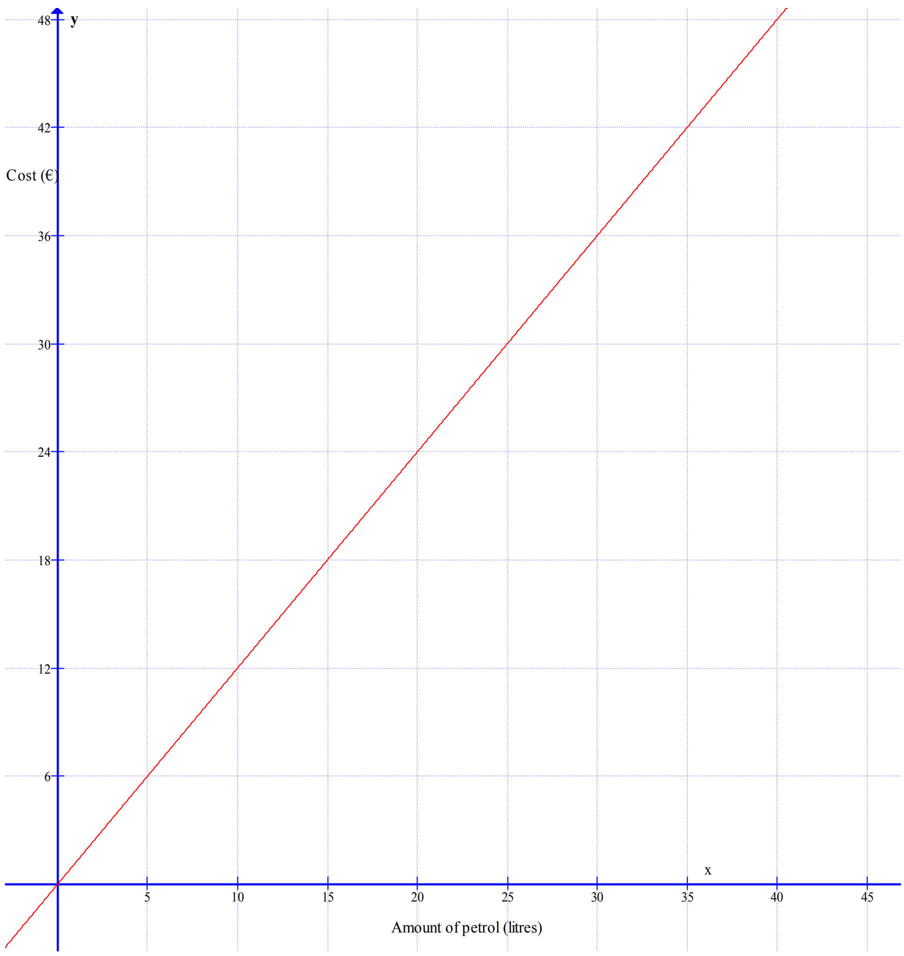
If the equation of a straight line has no y-intercept, the line goes through the origin and can be related to ratios.

Example:

When a car is filled with petrol, both the amount and the cost of the petrol are displayed on the pump. One litre of petrol costs 1.20 €. So, 2 litres cost 2.40 € and 5 litres cost 6 €.
The table below shows the different quantities of petrol as displayed on the pump:

Amount of petrol (litres)	5	10	15	20	25	30
Cost (€)	6	12	18	24	30	36

The information can be graphed, as shown below. Notice that for every 5 litres across the graph, the graph rises by 6 €. The ratio is $\frac{6}{5}$. This is the reason why the graph is a straight line. This idea can be used to solve a number of different types of problem.

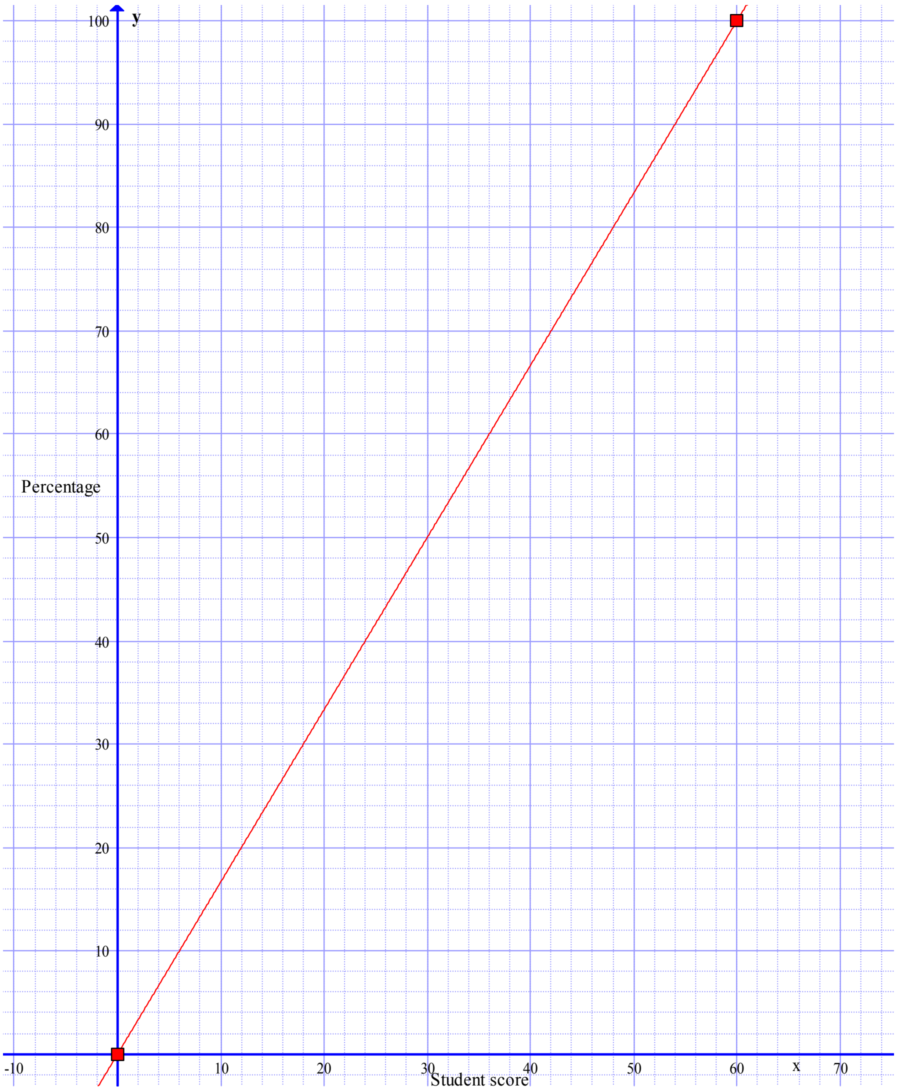


Exercise 8

Mr Smith wants to convert all students' scores in an English test to percentages. He uses the facts given below to help him to draw a conversion graph:

English score	0	60
Percentage	0	100

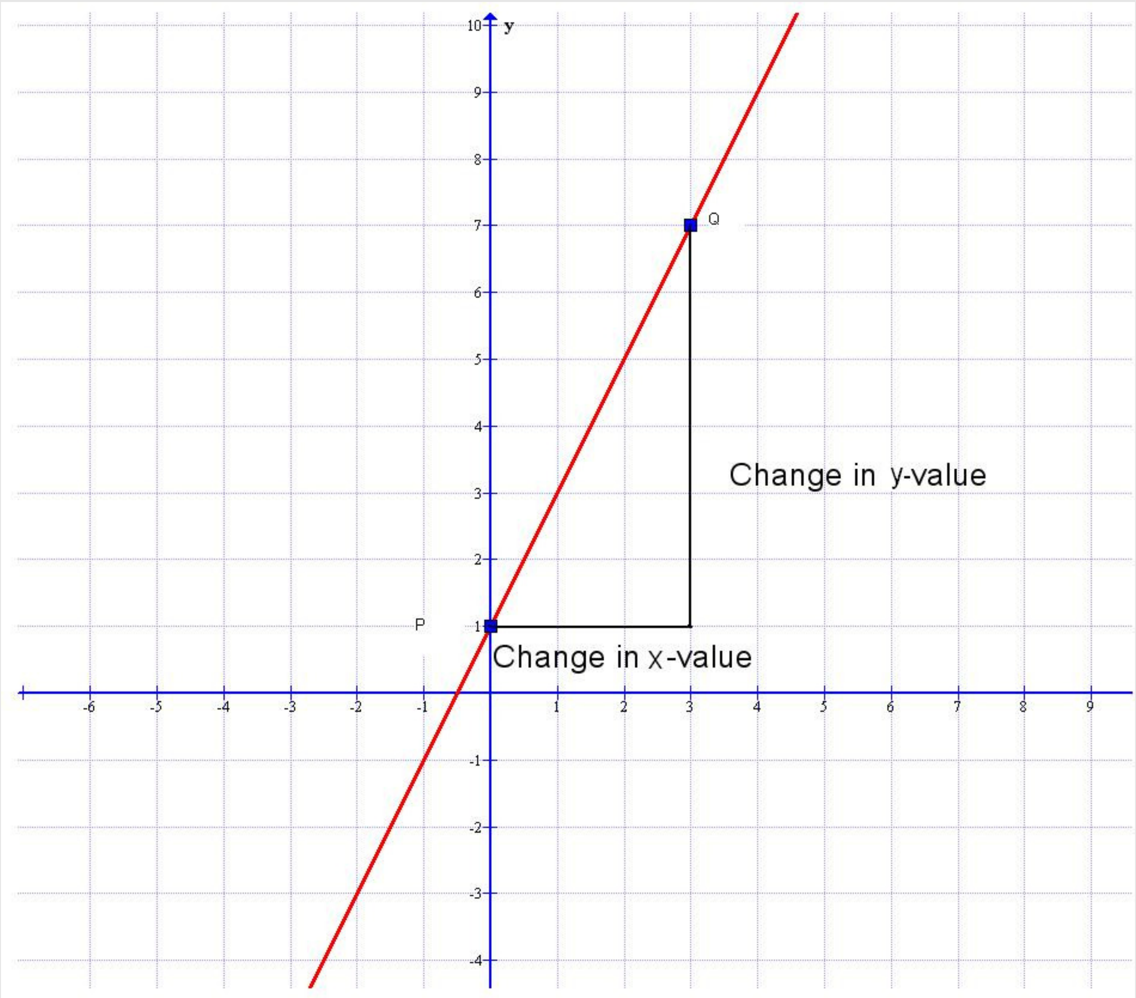
He uses the above two points to draw a linear graph.



What percentage does Stephanie get if she scores 30? What is Joe's score if his percentage is 63?

You can calculate the equation of a straight line from its graph. You just need to find the gradient and the y-intercept:

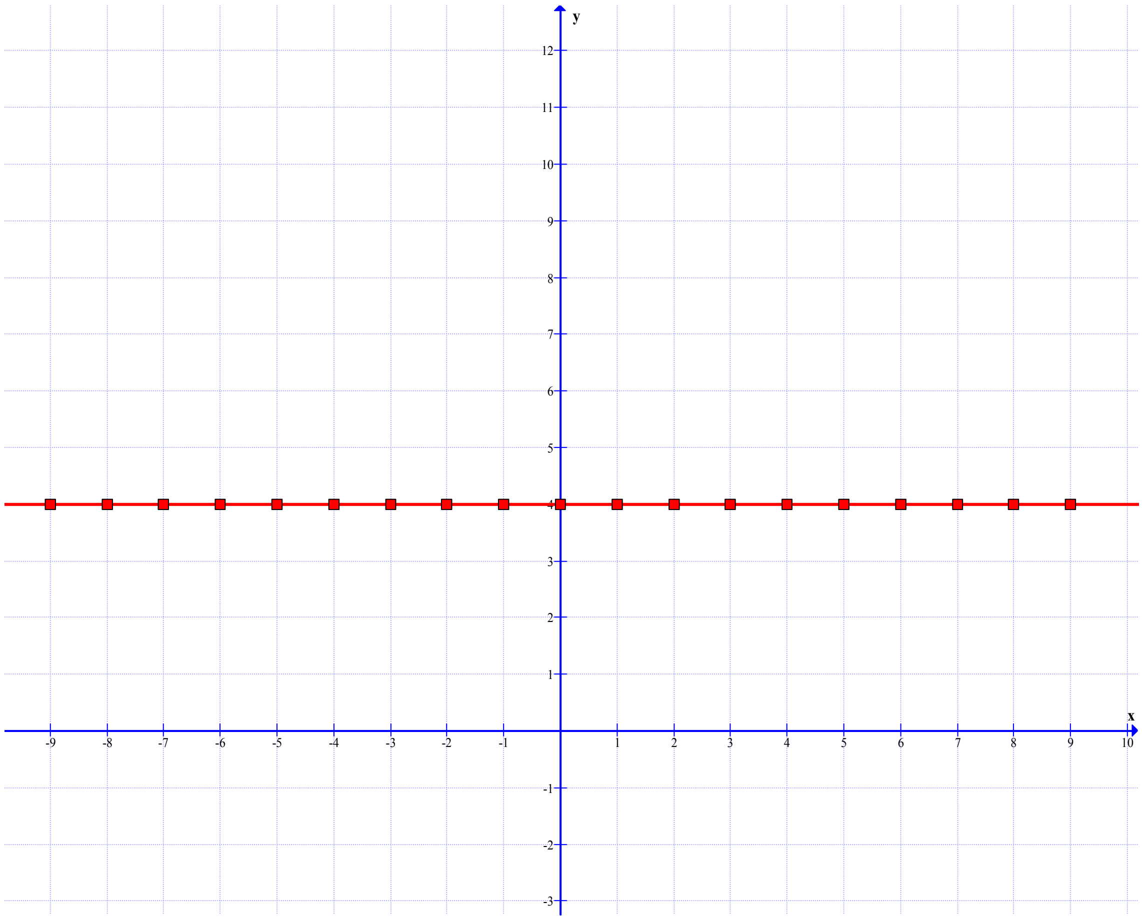
$$Gradient = \frac{increase\ in\ y - value}{increase\ in\ x - value}$$



There are some graphs which are very useful when we are working with functions. These are the **horizontal graphs**.

On this horizontal graph, the points are (-3,**4**), (-2, **4**), (-1, **4**), (0, **4**), (1, **4**), (2, **4**).

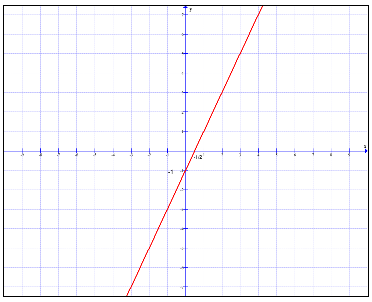
The y-coordinate is always 4, so the equation of the graph is y = 4.

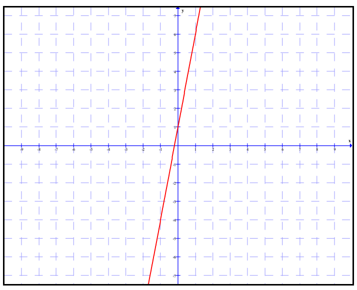


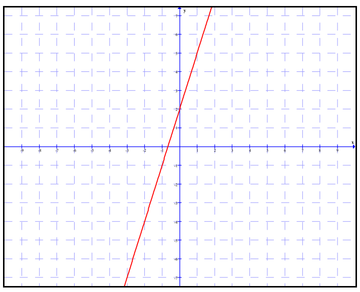
The equation of any horizontal line is always y = a, where a is a real number

Exercise 9

Match the equation cards with the graph cards, explaining how you made your choice.







y = 3x + 2	y = 2x – 1	y = 5x + 1

Exercise 10
Answer 'true' or 'false' to the following:

a) $y = 5x$ **is steeper than** $y = 3x$

b) $y = 2x$ **is parallel to** $y = 2x + 4$

c) $y = 4x$ **is steeper than** $y = x - 5$

d) $y = 2x$ **is parallel to** $y = x + 2.$

EXERCISES:

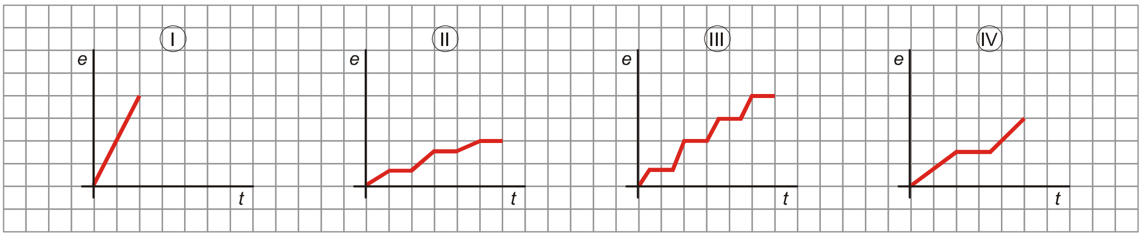
11 Which is the graph for each of the following situations? Indicate what the independent and dependent variable represents in each case:

a) Route made by an urban bus.

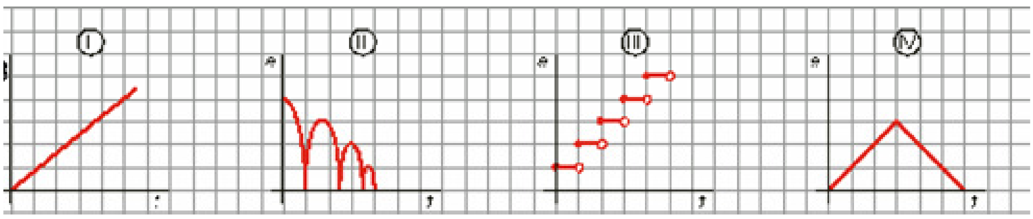
b) Bike ride round a park, stopping once to drink some water.

c) Distance made by a racing car in a circuit section.

d) A postman delivering the post.



12 Match every statement with the corresponding graph and indicate what the independent and dependent variable represent in each case:



a) Height of a ball that bounces constantly.

b) Cost of a phone call according to its time.

c) Distance to get home during a 30-minute walk.

d) Level of the water in an empty swimming pool when filling it.

13 Four friends are talking about their walk to school this morning.

- Jessica: I came by motorbike but I forgot an essay I have to hand in and I had to go back home. Then I ran as fast I could to go to the school.

- Brian: My mother brought me by car; but we found a traffic jam at the traffic lights situated half way and we were too late.

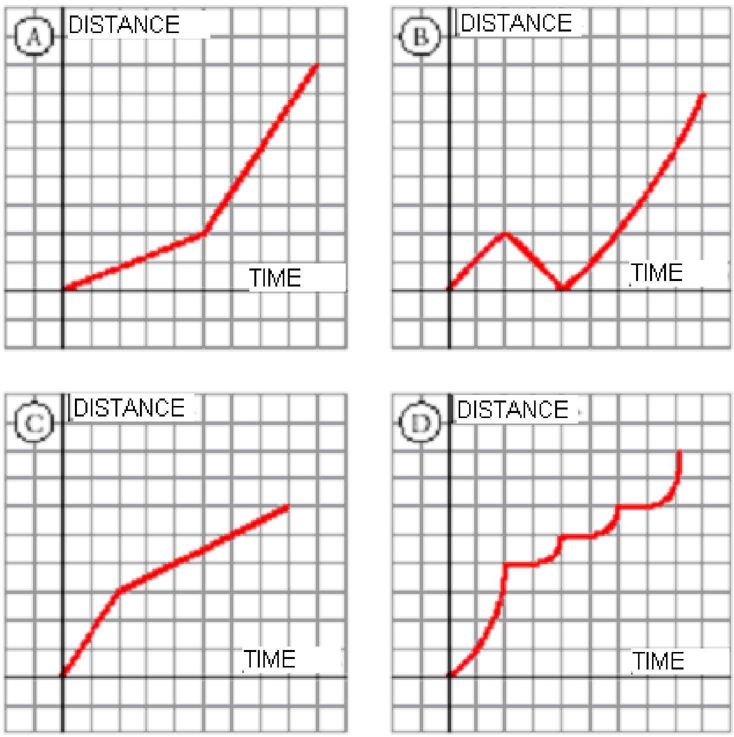
- Elena: I met a friend who attends a different school at the front door of my house. We walked together part of the way, and when we parted I had to hurry up. I was too late because of our walk.

- Andy: I left home very quickly because I had arranged to meet Maria and it was late. Then we walked together slowly.

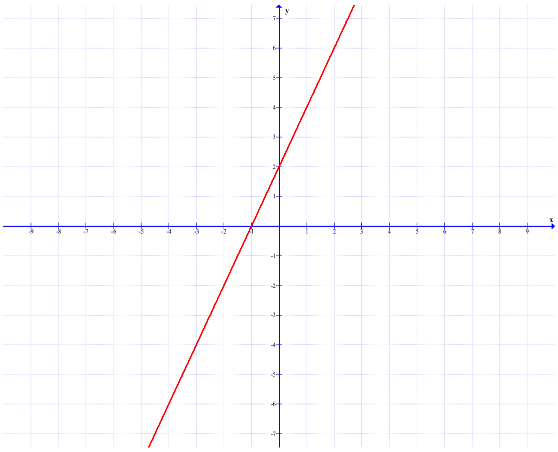
All of them go to the same secondary school, and each graph shows, in a different order, the path followed since they left home until the entrance to the school.

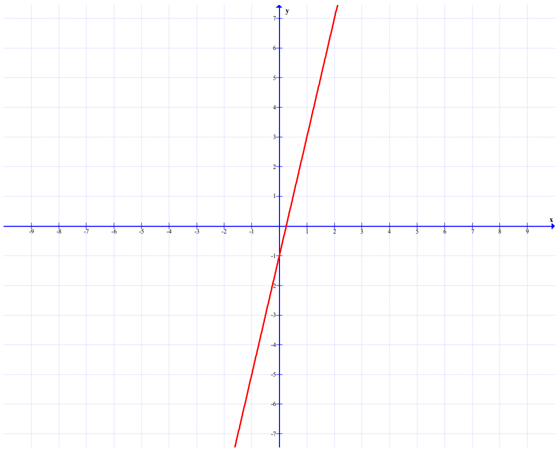
The same scale has been used in every graph.

Match each boy/girl with the graph that represents their walk to school.



14 Write the equations of the following straight-line graphs:





15 Write the equations of the following straight-line graphs:

a) Gradient is 5 and cuts the y-axis at (0, 1).

b) Gradient is $\frac{1}{2}$ and cuts the y-axis at (0, -1)

c) Parallel to y = 3x + 5 and cuts the y-axis at (0, 7)

d) Twice as steep as *y = x – 1* and cuts the y-axis as the same place as *y = x + 3*

e) Goes through the points (1, 5) and (2, 9)

16 Select the equations that produce horizontal graphs and then make a sketch of them indicating where they cut the y-axis:

a) $y = 3x^2$ b) y = - 4 c) y = 2x – 6 d) y = 2 e) x = - 1

9 Statistics

Keywords:			
Data	frequency table	bar chart	pie chart
interval	tally	pictogram	population
pyramid	temperature and Rainfall chart		mean
median	mode	average deviation	

Previous ideas

a) Sort data into discrete, continuous and categorical.
b) Choose the most appropriate graph to display data: bar chart, pie chart, pictogram,

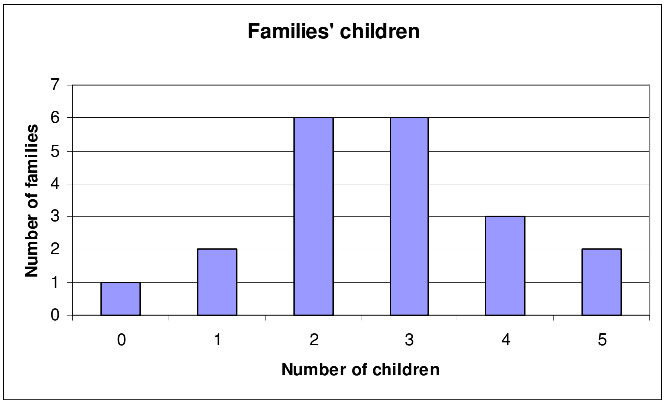
1. Constructing a frequency table

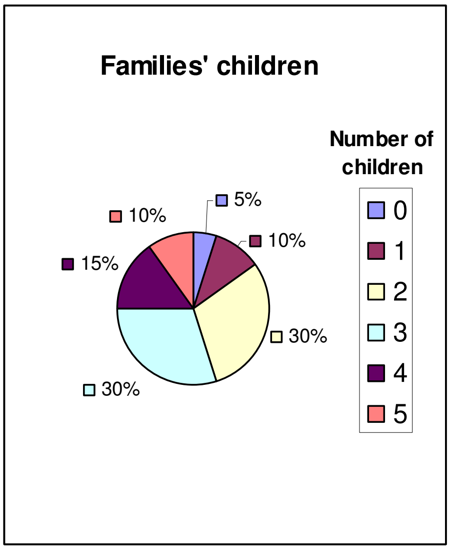
Example 1 Twenty families are asked about how many children they have. These are the answers:

3	3	4	1	2	3	2	5	1	0
2	2	3	2	4	2	5	3	4	3

Data	Frequency
0	1
1	2
2	6
3	6
4	3
5	2

Now, we can draw a bar chart or a pie chart:





Histogram:

The histogram is used for variables whose values are numerical and measured on an interval scale.
A histogram divides up the range of possible values in a data set into classes or groups. For each group, a rectangle is constructed with an area proportional to the frequency, (if the bars have equal width the height of each bar corresponds to the frequency).
Notes:
A vertical bar graph and a histogram differ in these ways:

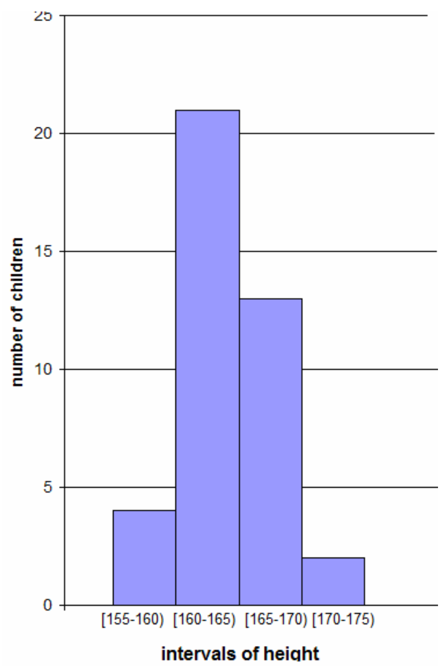
In a histogram, frequency is measured by the *area* of the column.
In a vertical bar graph, frequency is measured by the *height* of the bar.

Example 2 Mario decided to collect data about the height of his partner in the school. These are the data of 40 children (in cm):

163	167	165	159	164	168	161	164	163	164
165	163	167	165	164	164	168	161	164	165
163	164	170	160	157	167	165	172	165	167
164	164	168	161	164	163	164	155	158	162

It is useful to make a grouped frequency table for this case:

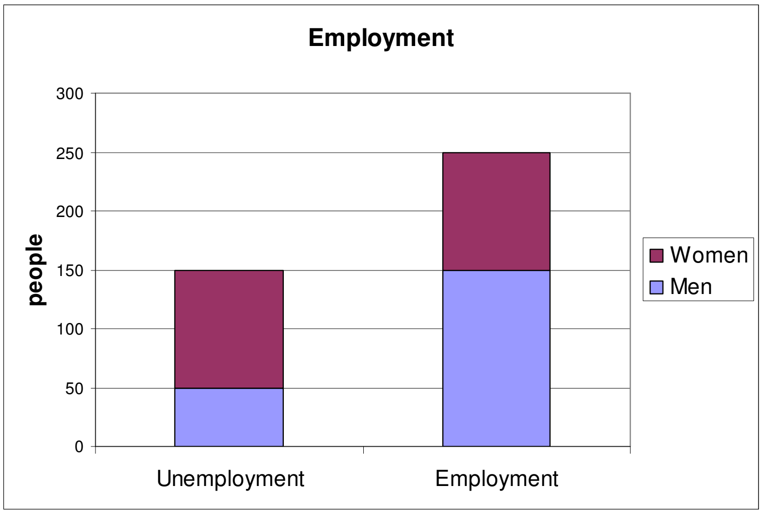
Interval	Tally	Frequency
[155-160)	IIII	4
[160-165)	~~IIII~~ ~~IIII~~ ~~IIII~~ ~~IIII~~ I	21
[165-170)	~~IIII~~ ~~IIII~~ III	13
[170-175)	II	2



Example 3 These are the data about employment in a town.

	Unemployment	Employment	Total
Men	50	150	200
Women	100	100	200
Total	150	250	400

We draw a composite bar chart to display the data.



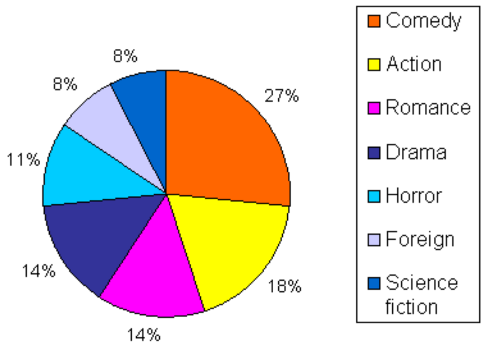
In a composite bar chart you can compare totals easily, but is hard to see the separate data.

2. Interpreting diagrams

Pie Charts

A pie chart is a circle graph divided into sectors, each displaying the size of some related piece of information. Pie charts are used to display the sizes of parts that make up some whole.

Example 4 Favorite movie genres in Mrs. Lucia's Film class



In the pie chart above, the legend is made properly and the percentages are included for each of the pie sectors. However, there are too many items in the pie chart to quickly give a clear picture of the distribution of movie genres. If

there are more than five or six categories, consider using another graph to display the information.

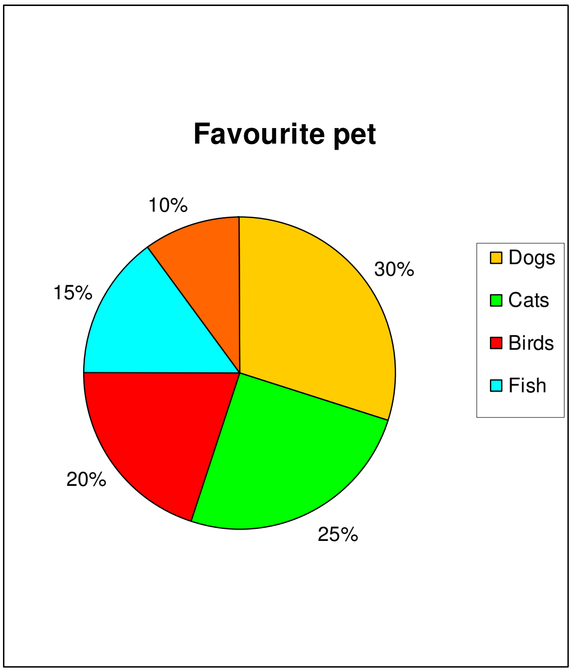
a) Suppose 100 students were in all. Then there were 27 students that prefer comedy movies, 18 that prefer action movies, 14 romances, 14 drama too, 11 horror, 8 foreign and 8 science fiction.
b) Suppose 300 students. Then there were 71 students that prefer comedy movies, …

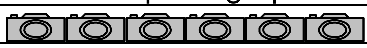
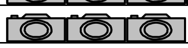
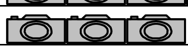
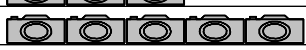
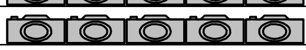
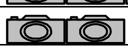
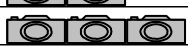
Exercise: In my high school there are 200 students. They were asked about their favourite pet. See the figure and answer the questions:

How many people answer "dogs"? And "cats"?

Pictogram

These are the number of photographic cameras sold the last year
10 photographic cameras 

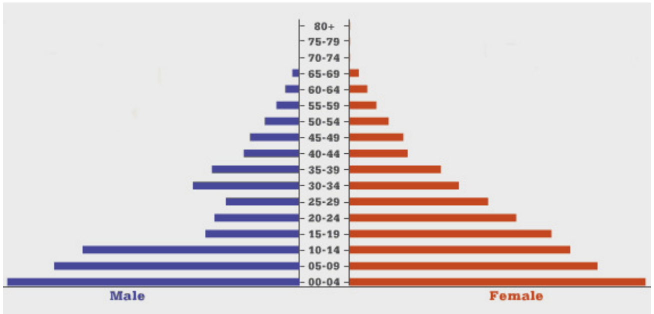


	Cameras photographic
January	
February	
March	
April	
May	
June	
July	
August	
September	
October	
November	
December	

How many cameras had been sold every month? Which one was the best month for the salesman? Why do you thing was this month?

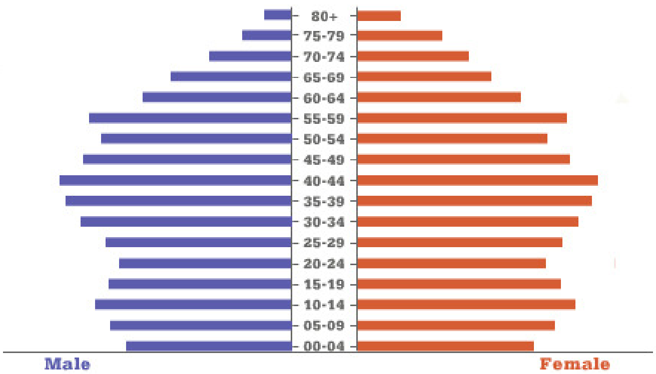
Population pyramid

Example 1



This population pyramid has a very wide base, showing that this country has a high birth rate.
This is likely to be the population pyramid of a less developed country and because of a high infant death.
There is a sharp indent in the male side from 15 years upwards. This could be because of a conflict or migrated to other countries to get jobs.
There is a low life expectancy.

Example 2

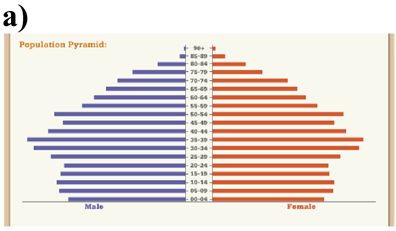


The largest grouping is between 30 and 45.
The problem in the future will be when there few people paying taxes to support the pensions.
There are large numbers of people living into their 80s.

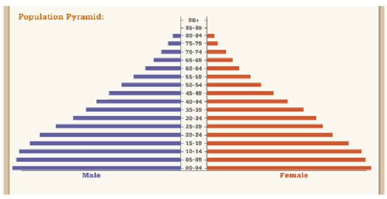
Exercise 1

See these three population pyramids:

a)



b)



Could you say which one is from Mexican or UK?

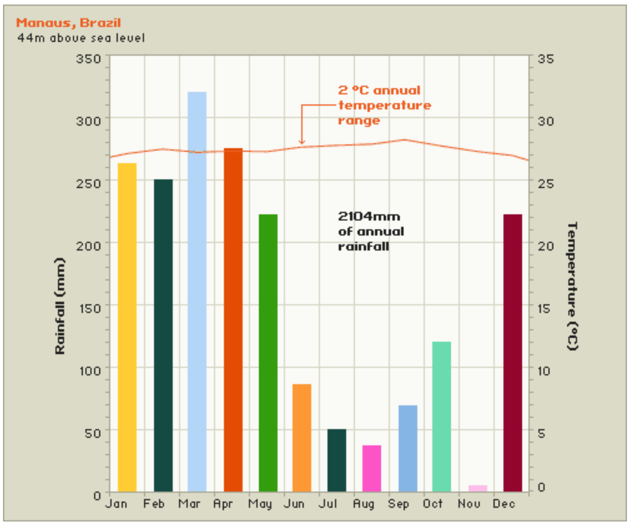
Climate Data: Temperature and Rainfall Charts

Example

The graphic shows average rainfall and temperature in Manaus, Brazil, in the Amazon rainforest.

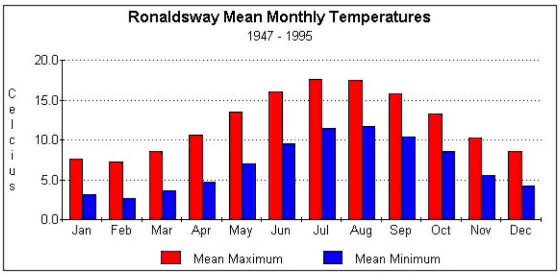
Total rainfall is 2104mm per year, most of it during the rainy season from December to May.

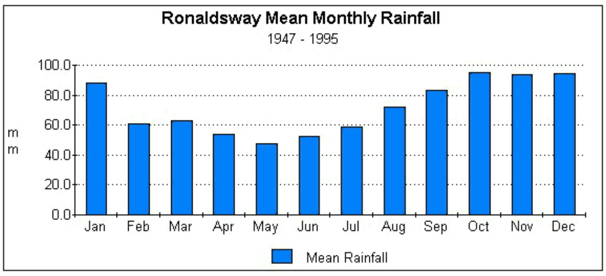
Notice how much the rainfall varies over the year: the highest monthly rainfall is in March with over 300mm, while the lowest is in August with less than 50mm. Meanwhile the temperature only varies by 2℃.

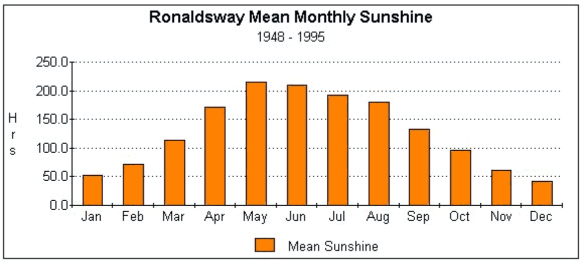


Exercise 2

These are the data from Ronaldsway airport in Isle of Man (between England, Scotland and Gales):







Could you explain each other?

What about your city? Are similar?

3. Parameter statistics

Mean

The mean of a list of numbers is also called the average. It is found by adding all the numbers in the list and dividing by the number of numbers in the list.

Example 1 Find the mean of 10, 11, 7, and 8 to the nearest hundredth.

$$\text{mean} \equiv \bar{x} \equiv \frac{10+11+7+8}{4} = 9$$

Median

The median of a list of numbers is found by ordering them from least to greatest. If the list has an odd number of numbers, the middle number in this ordering is the median. If there is an even number of numbers, the median is the sum of the two middle numbers, divided by 2. Note that there are always as many numbers greater than or equal to the median in the list as there are less than or equal to the median in the list.

Example 2 My marks in Maths were 7, 10, 8, 10, 7 last year. Find the media. Placed in order, the age's marks were 7, 7, 8, 10, 10. The number 8 is the median.

Imagine my marks were 7, 10, 8, 10, 7 and 7. So the middle numbers are 7 and 8, which are the 3th and 4th.
The median is the average of these two numbers:

$$\frac{7+8}{2} = \frac{15}{2} = 7.5$$

Mode

The mode in a list of numbers is the number (or numbers) that occurs most often.

Example

The students in Bjorn's class have the following ages: 5, 9, 1, 3, 4, 6, 6, 6, 7, 3. Find the mode of their ages. The most common number to appear on the list is 6, which appears three times. No other number appears that many times. The mode of their ages is 6.

Average deviation

Deviation is a measure of difference for interval and ratio variables between the observed value and the mean.
Average deviation is calculated using the absolute value of deviation (it is the sum of absolute values of the deviations) divided by the number of data.

The ages of my cousins are 10, 15, 12, 13, 10, 12
We calculate the mean:

$$\text{Mean} \equiv \bar{x} = \frac{10+15+12+13+10+12}{6} = 12$$

Data	10	15	12	13	10	12
Absolute value of deviation	2	3	0	1	2	0

$$\text{Average deviation} \equiv \frac{2+3+0+1+2+0}{6} = 1.33$$

Exercises3

3.1 The tallest 4 trees in a park have heights in meters of 40, 52, 50, 55. Find the median of their heights.

3.2 Find the mean, median, mode and average deviation for the following data:

a) 10, 12, 13, 12, 13, 10, 14 and 13.

3.3 Twenty families are asked about how many children they have. These are the answers:

3	**3**	**4**	**1**	**2**	**3**	**2**	**5**	**1**	**0**
2	**2**	**3**	**2**	**4**	**2**	**5**	**3**	**4**	**3**

a) Find the mean, median, mode and average deviation.

b) Make the frequently table. Can you use it for a)

10 Similarity

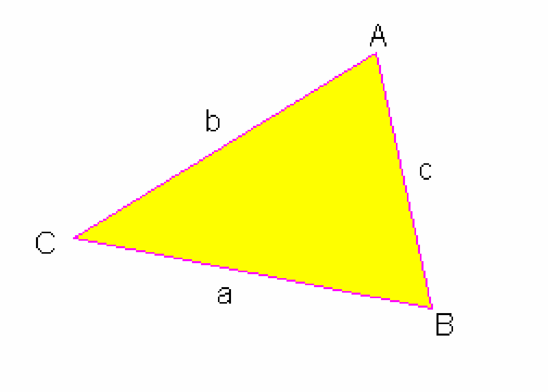
Keywords:

Similarity **Scale Thales position** **similarity criterion**

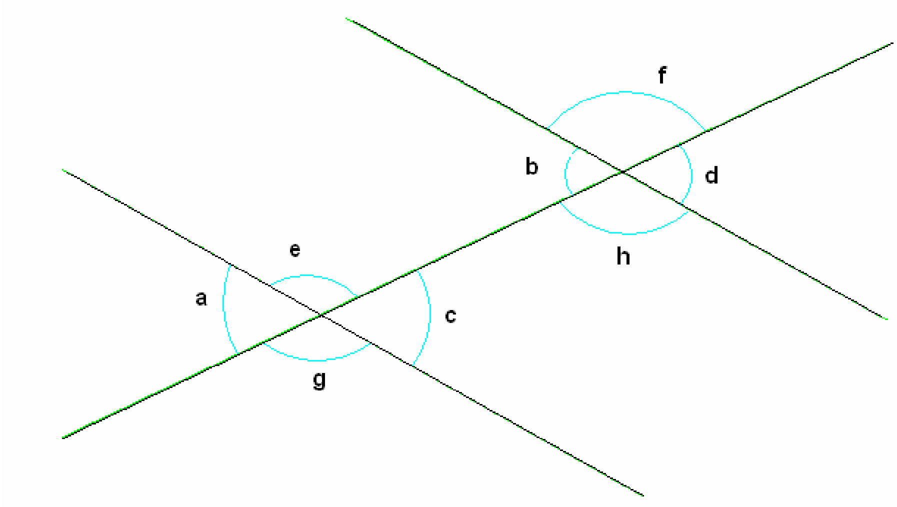
Previous ideas

1 Triangle

A triangle is a three-sided polygon. We use capital letters to name the vertices and lower case letters to the opposite sides like this



2 Angles in two intersecting lines crossed by two parallels

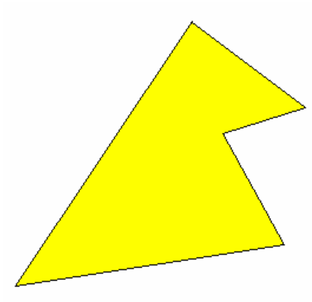


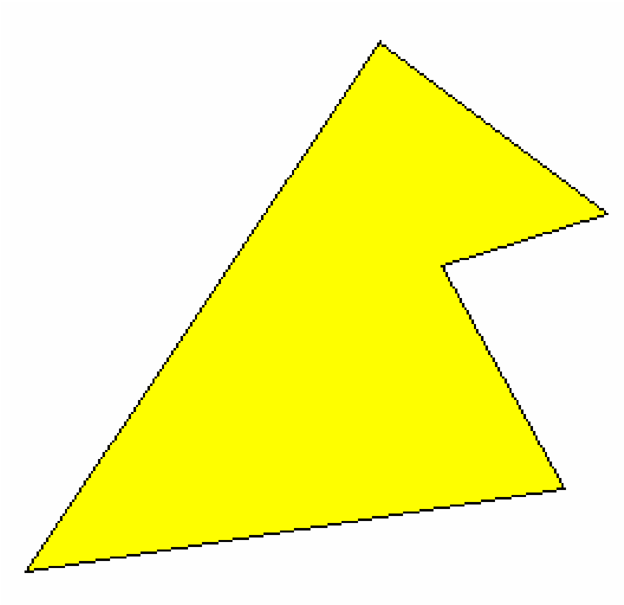
Alternate Interior Angles c = b and e = h
Alternate Exterior Angles f = g and a = d
Corresponding Angles e = f and a = b
f = h or b = d, for example, are **Vertical Angles.**

1 Similar shapes

Two geometrical objects are similar when one is the result of enlarging or shrinking the other, corresponding angles are equal and the length of corresponding segments are in the same ratio

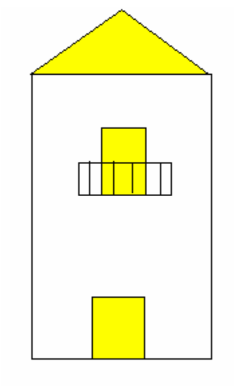
Example These two shapes are similar

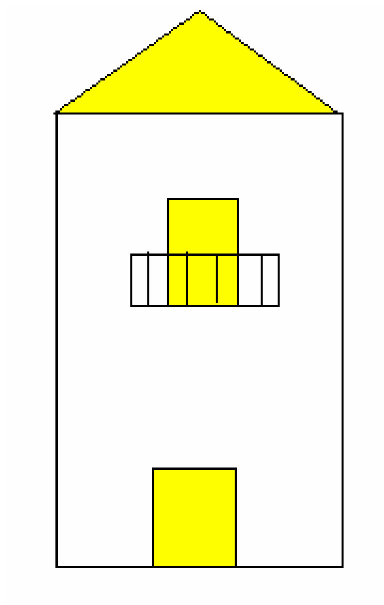




Exercise 1 Use a protractor to check that corresponding angles are equal and use a ruler to find out which is the ratio between the corresponding sides of the shapes.

Exercise 2 Find out if these two pictures are similar





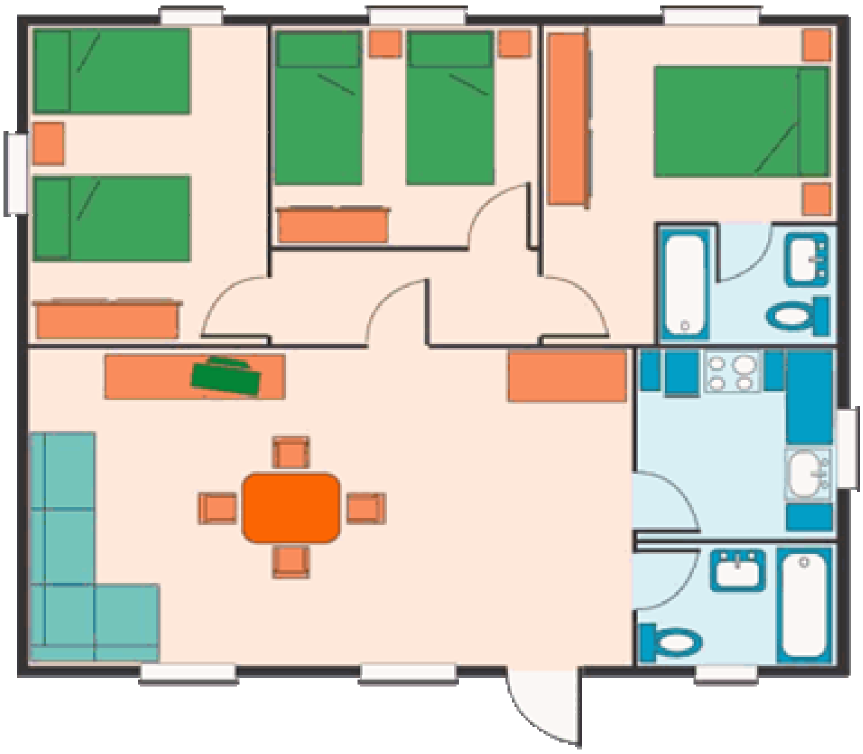
The height of the door on the left is 2 m. Which is the height of the door on the right?

Calculate the real height of the house on the left.

2 Maps

Maps are representations of an area. There is a scale, the ratio between the real objet and the map, and then we can know the real distance between two points.

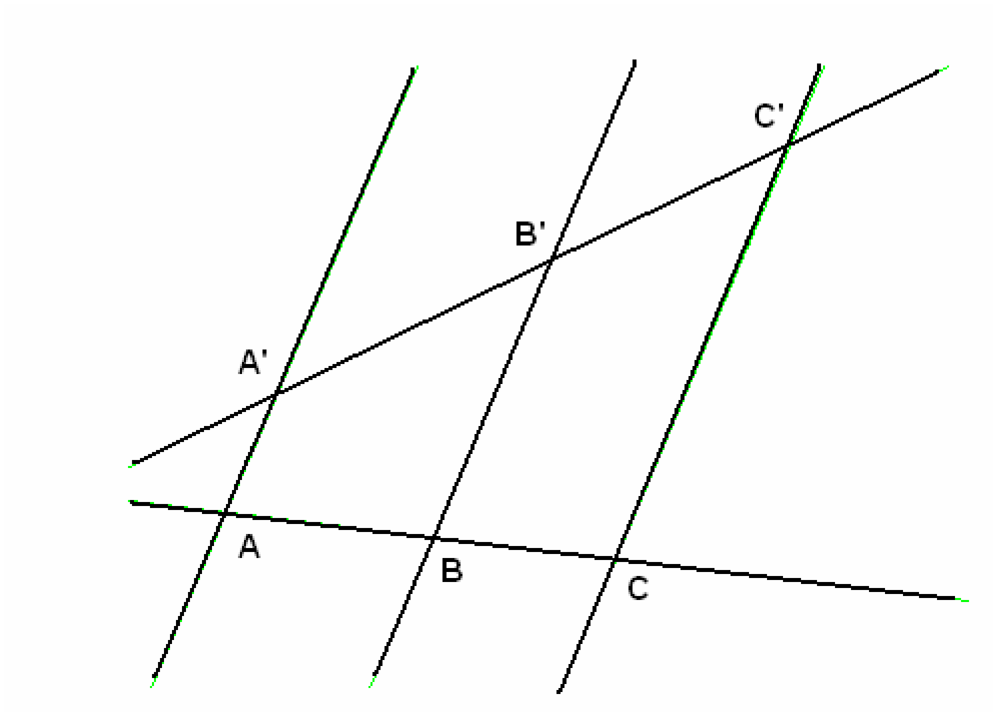
Example in this map the scale is 1:100 it means that one cm of the map is 1 m in the house



Exercise 2 Find the dimensions of each room of this house. Calculate the size of the flat in m^2

2 Thales Theorem (Intercept theorem)

If two non parallel lines intersect with parallel lines, the ratios of any two segments on the first line are equal to the ratios of the corresponding segments on the second line



That is:

$\frac{\overline{AB}}{\overline{BC}} = \frac{\overline{A'B'}}{\overline{B'C'}}$, you can also say that $\frac{\overline{AB}}{\overline{A'B'}} = \frac{\overline{BC}}{\overline{B'C'}}$

Exercise 3 Calculate the ratio (use a ruler) $\frac{\overline{AB}}{\overline{BC}}$ **or** $\frac{\overline{A'B'}}{\overline{B'C'}}$ **of the picture above.**

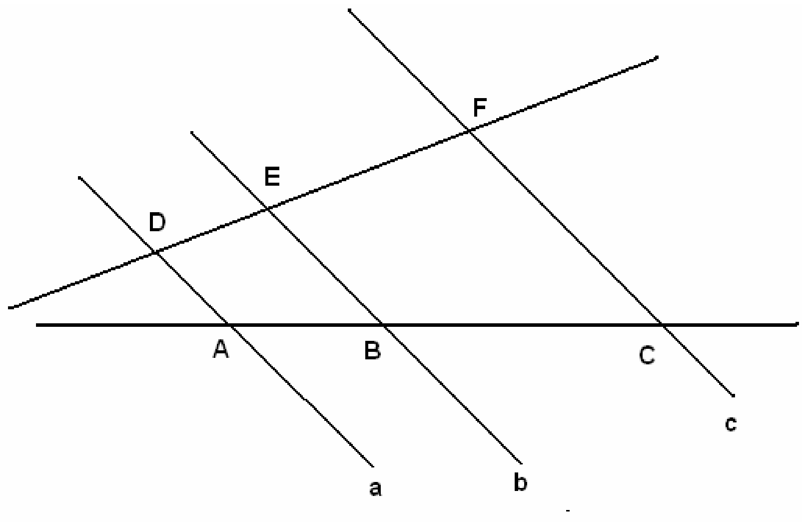
Exercise 4 Consider that lines a, b and c are parallel lines and

$\overline{AB} = 2\ \text{cm}$**,**

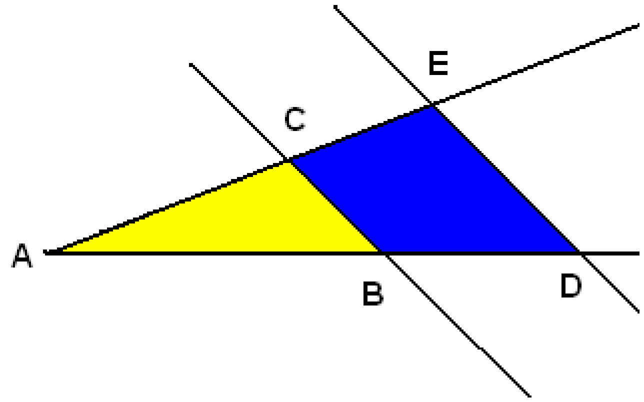
$\overline{EF} = 2.8\ \text{cm}$ **and**

$\overline{BC} = 3.5\ \text{cm}$**.**

Calculate $\overline{DE}$

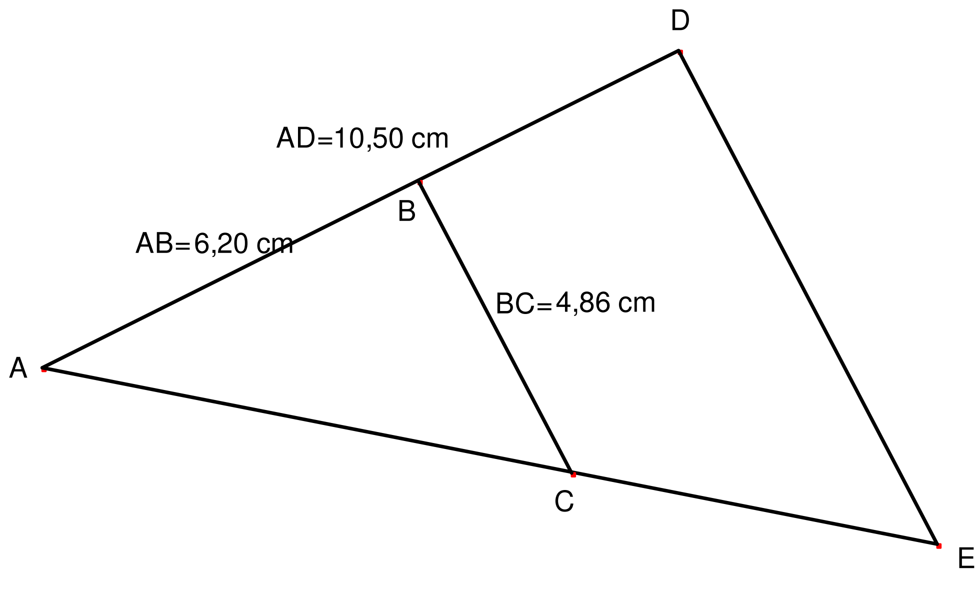


3 Triangles put in the Thales position. Similar triangles



If we use the Thales theorem in the drawing above in which the non parallel lines intercept at A, the triangle ABC is inside and fitted on the triangle ADE, we say that these two triangles are in the Thales position and they are similar because corresponding angles are equal and corresponding sides are in proportion

Exercise 5 Use the Thales theorem to find DE in the picture below



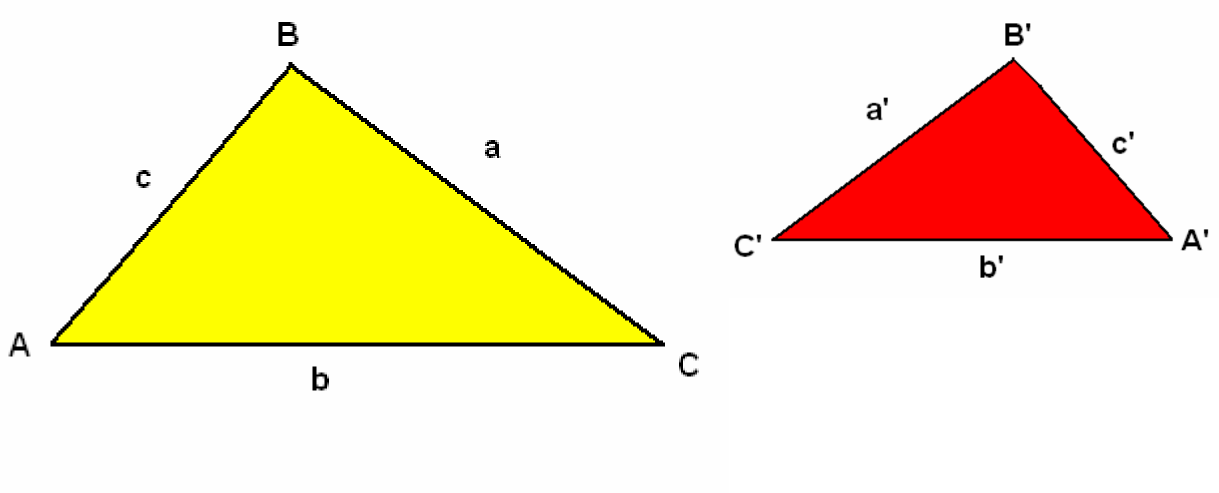
Use a ruler to measure the real lengths and a protractor to measure the angles in the two triangles. Check that the triangles ABC and ADE have the same angles and their sides are in proportion. What can you say about these triangles?

3 Similar triangles, similarity criteria

To be sure that two triangles are similar, we do not need to see if all the three corresponding angles are equal and corresponding sides are in proportion. We can assure that they are similar with fewer conditions. These sets of conditions are called similarity criteria.

3.1 First similarity criterion

Two triangles ABC and A'B'C' are similar if $\hat{A} = \hat{A}'$ and $\hat{B} = \hat{B}'$ because then $\hat{C} = \hat{C}'$. Remember that $\hat{A} + \hat{B} + \hat{C} = \hat{A}' + \hat{B}' + \hat{C}' = 180º$.



3.2 Second similarity criterion

Two triangles ABC and A'B'C' are similar if all the three sides are in proportion that is $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$, this is true because these triangles could be put in the Thales position on any vertex.

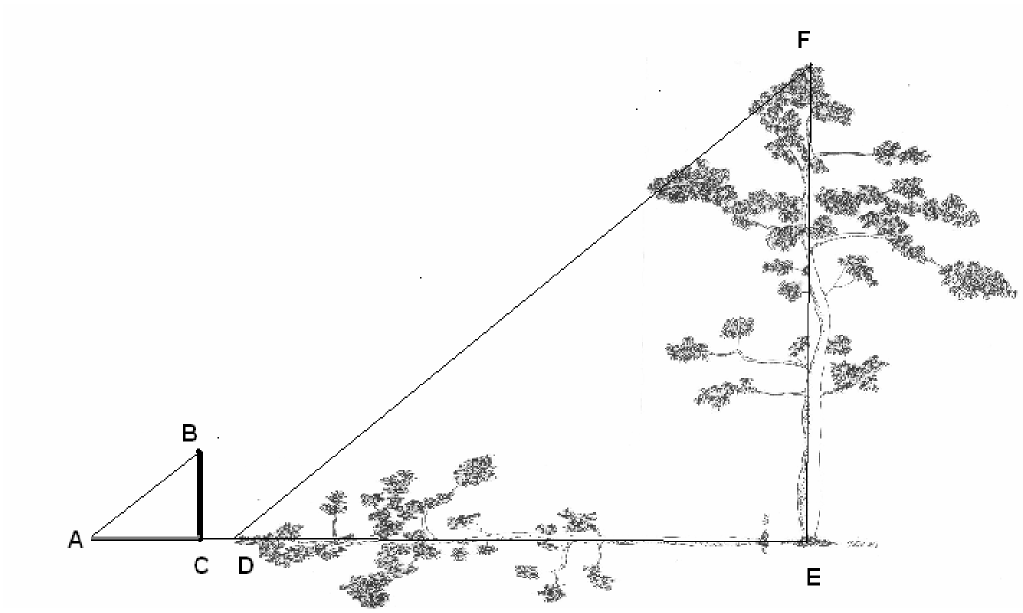
3.2 Third similarity criterion

Two triangles ABC and A'B'C' are similar if $\hat{A} = \hat{A}'$ and $\frac{b}{b'} = \frac{c}{c'}$, this is like that because these triangles could be put in the Thales position on the vertex A.

Exercises 6

Exercise 6.1

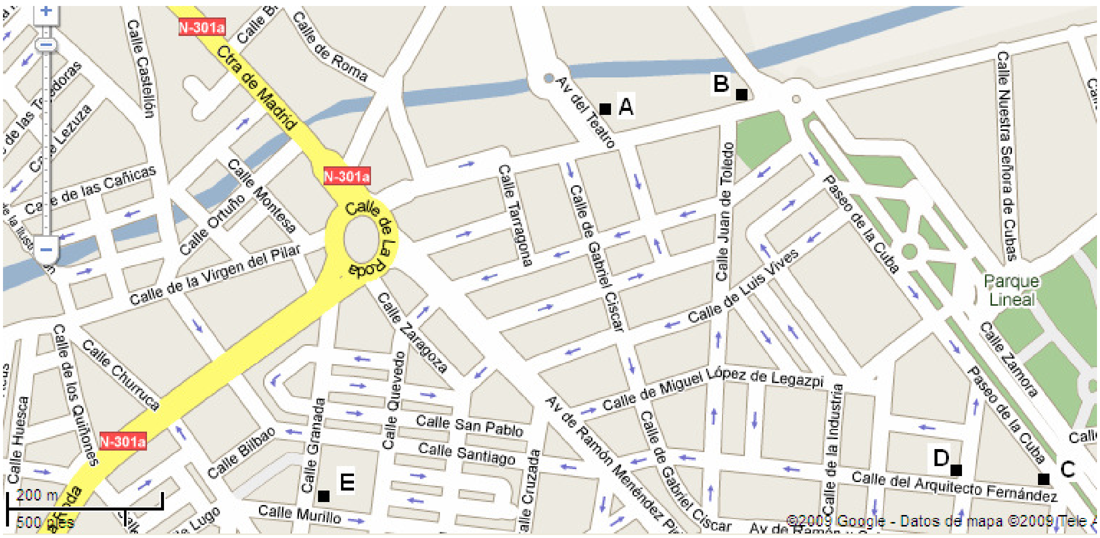
Measuring the length of the shadow of a stick, we can calculate the height of a tree. Calculate the height of the tree from the picture below considering that the length of the stick is 1.25 m its shadow is 1.52 m and the shadow of the tree is 6.3 m



6.2 Look at the scale of this map and calculate:

a) **The distance covered by a person that goes from our school in A to their house in Arquitecto Fernández (D) following the route ABCD**

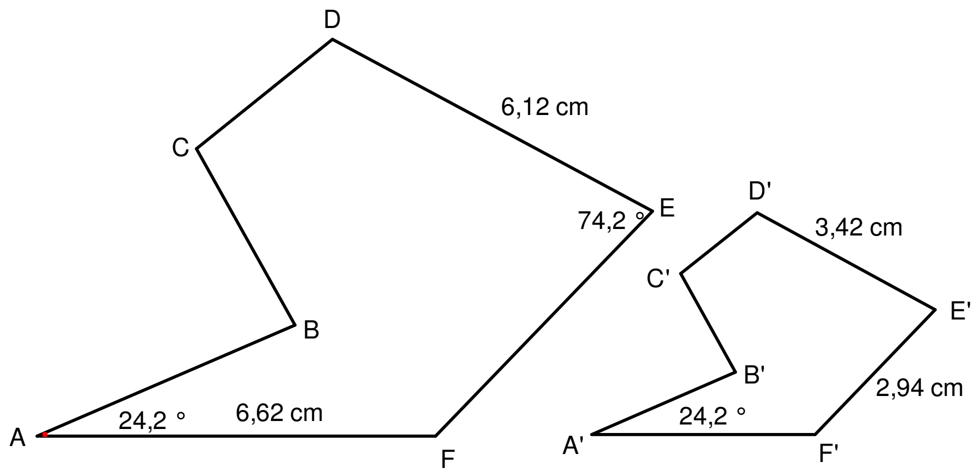
b) **The shorter distance that has to be covered if we want go from A to E**



6.3 The two polygons are similar. Find out:

a) FE **b)** Ê' **c)** A'F'

d) What is the ratio between every pair of sides?

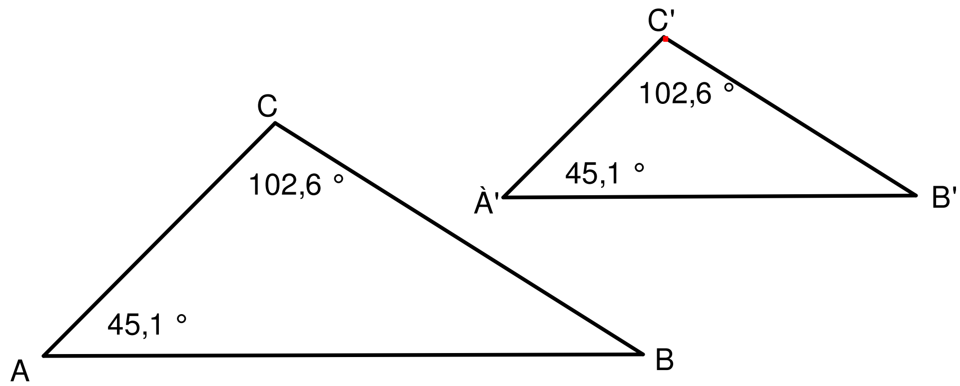


6.3 The three sides of a triangle ABC are $a = 2\text{ cm}$ **,** $b = 2.7\text{ cm}$ **and** $c = 3.4\text{ cm}$ **. The triangle A´B´C´ is similar and** $a' = 2.5\text{ cm}$ **. Which are the values of the other two sides? Draw these triangles using a ruler and a compass.**

6.4 The triangles ABC and A`B`C` have $\hat{A} = \hat{A}' = 45{,}1º$ **and** $\hat{C} = \hat{C}' = 102{,}6º$

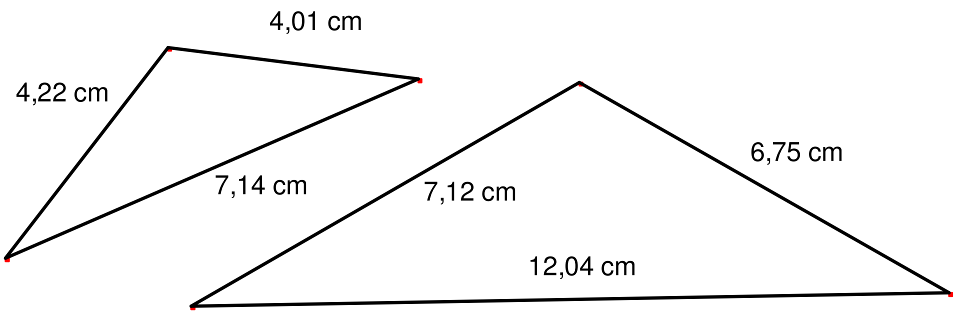
a) Check that they are similar triangles

b) Find the value of the angle $\hat{B}$

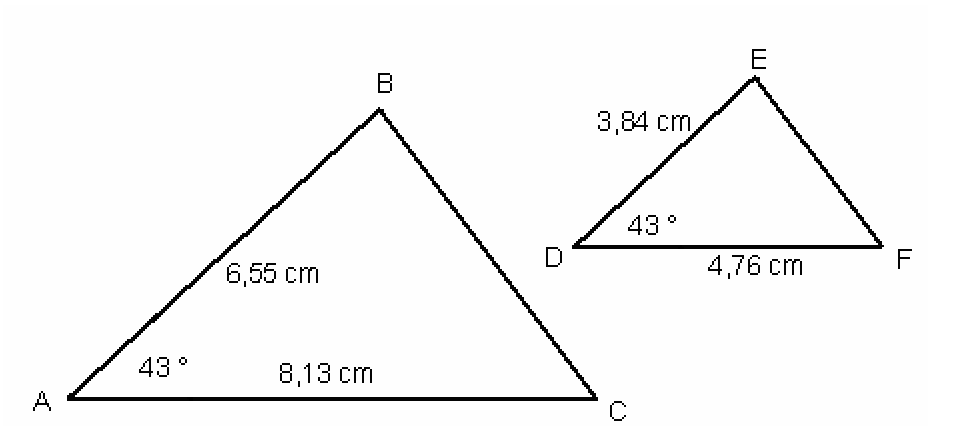


c) Calculate the similarity ratio

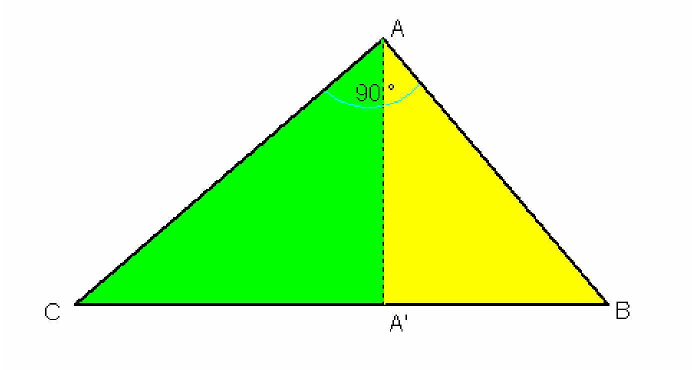
6.5 Prove without measuring that these triangles are similar. Check that their corresponding angles are equal.



6.6 Prove without measuring these triangles are similar. Check that corresponding angles are equal and sides a and d are in proportion with the other corresponding sides



6.7 Prove that the triangles AA`C and AA`B are similar. Which are the corresponding sides of these triangles?



11 3-D shapes

Keywords:					
Cuboid	cube	prism	pyramid	base	apex
slant height	trunk		cylinder	cone	sphere

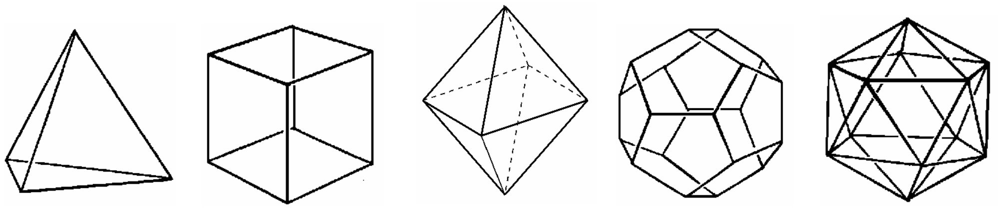
Remember

Polyhedrons

Polyhedrons are geometric solids whose faces are formed by polygons

Components: **Faces, Edges, Vertices, Dihedron angle**

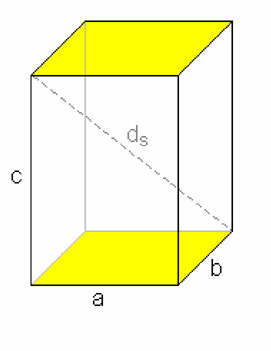
Regular polyhedrons:

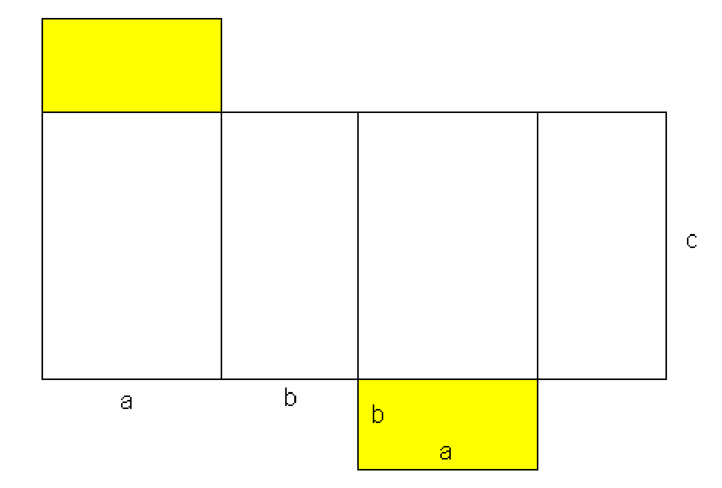


Tetrahedron Cube Octahedron Dodecahedron Icosahedron

1 Cuboid

In cuboids, faces are rectangles. They have 6 faces, 12 edges and 8 vertices





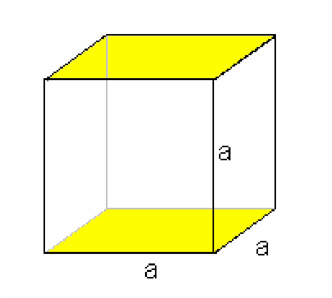
1.1 Area and volume of a cuboid

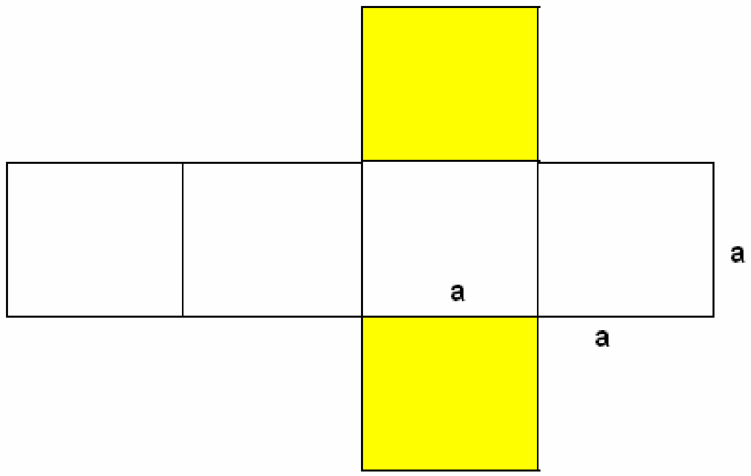
If we name the edges of the cuboid a, b and c, its area is:

$A = 2ab + 2ac + 2bc = 2(ab + ac + bc)$ And the volume is $V = a \cdot b \cdot c$

1.2 Area and volume of a cube

The special case of cuboids in which all the faces are squares is the **cube.**





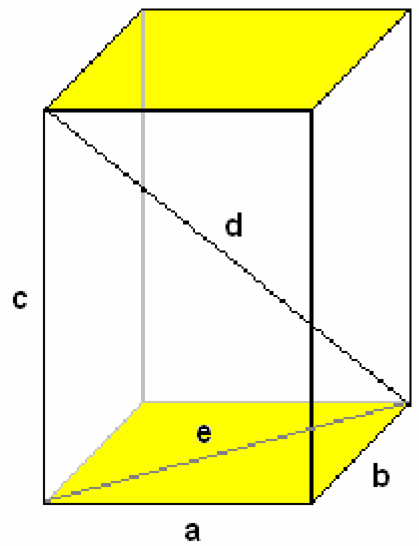
The area of a cube is $A = 6a^2$

The volume is $V = a^3$

1.3 Diagonal of a cuboid

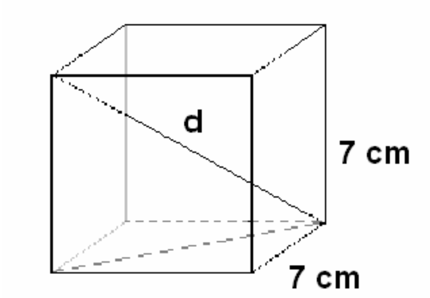
In this cuboid the triangle with sides **a**, **b** and **e** is a right triangle, so using the Pythagorean Theorem we can say $e^2 = a^2 + b^2$, but the triangle with sides **c**, **e** and **d** is a right triangle as well in which d is the hypotenuse and $d^2 = e^2 + c^2$, that is: $d^2 = a^2 + b^2 + c^2$ And finally

$d = \sqrt{a^2 + b^2 + c^2}$

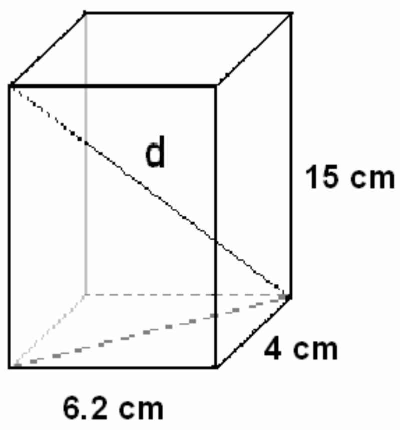


Exercise 1 Find the area, the volume and the diagonal of each solid shown below:

a)

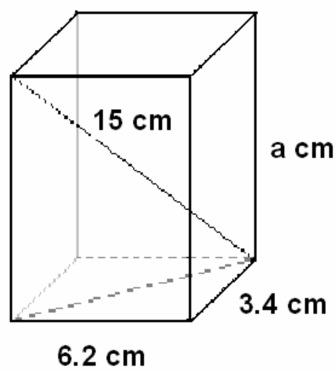


b)

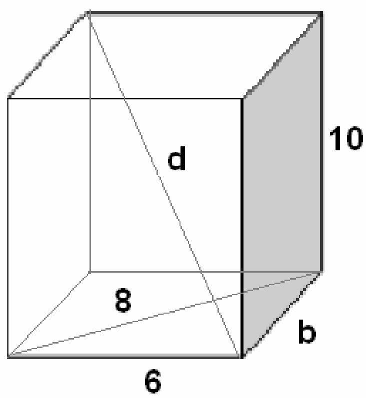


Exercise 2 Find the area, the volume and the missing length of these cuboids:

a)

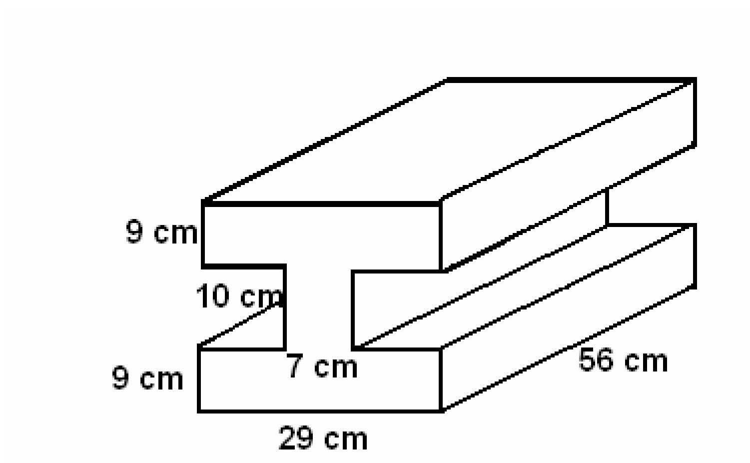


b)

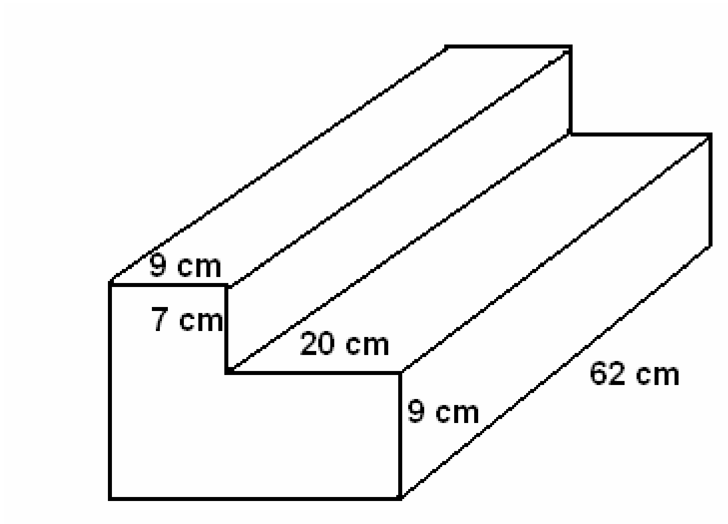


Exercise 3 Find the area and the volume of each solid shown below:

a)



b)



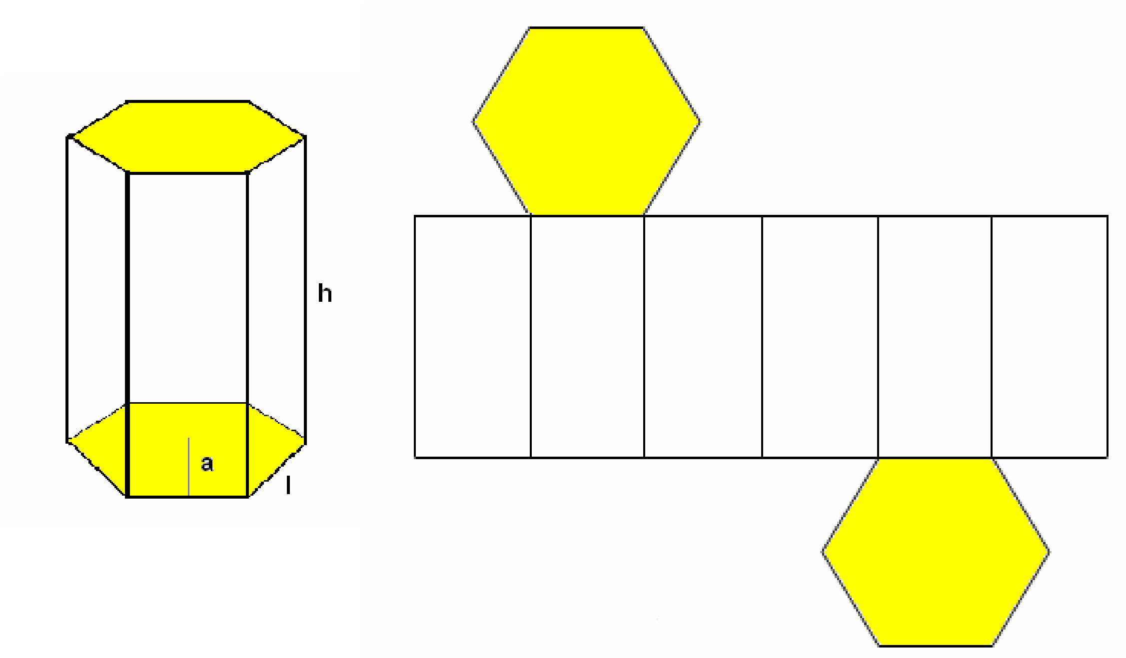
2 Prisms

They have a constant cross-section, side faces are parallelograms

The distance between the two bases is the **height** of the prism.
Depending on the polygons of the bases they can be:

Triangular prisms, square prisms, pentagonal prisms, etc.

This is an hexagonal prism



2.1 Area and volume of prisms

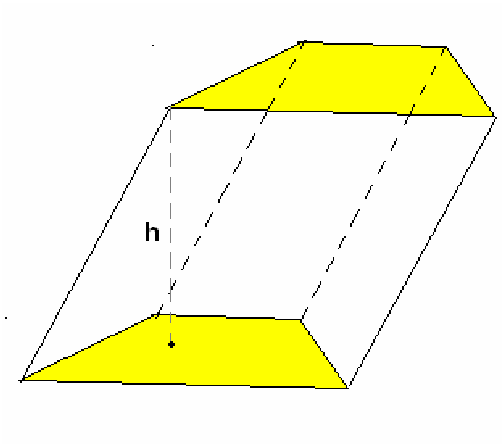
The area of a prism is found by adding the areas of its faces, if we call **l** the side of the base, **a** the apothem and **h** the height of the prism, the total area is $A = n \cdot l \cdot a + n \cdot l \cdot h$

Since the area of each base, which is a regular polygon of **n** sides, is $A_b = \dfrac{n \cdot l \cdot a}{2}$ and we have **n** rectangles each one of width **l** and height **h**

The volume is the area of the cross-section (base) multiplied by the height

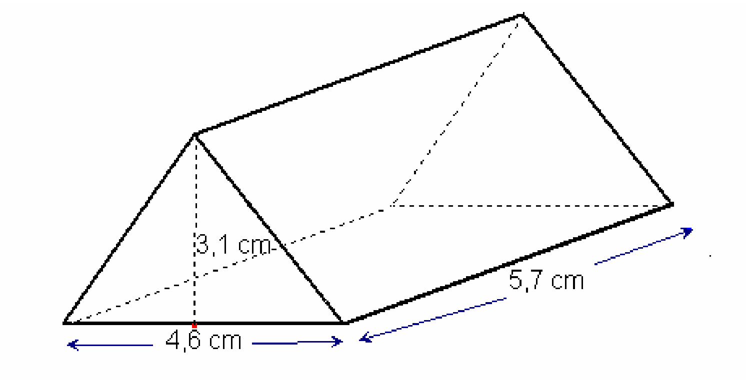
$V = \text{base area} \times \text{height}$

When the prism is oblique the formula is the same but the height is not the length of the edge.

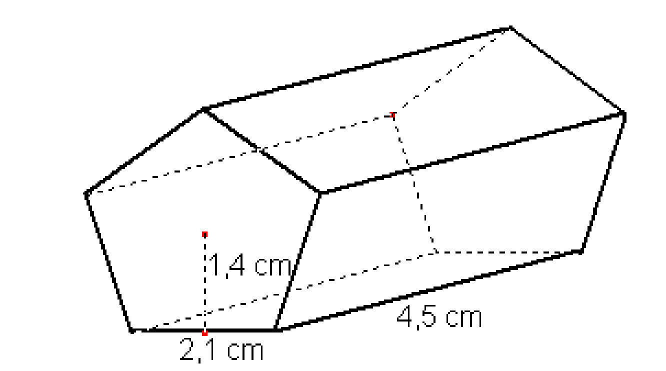


Exercise 4 Find the area and the volume of each prism shown below:

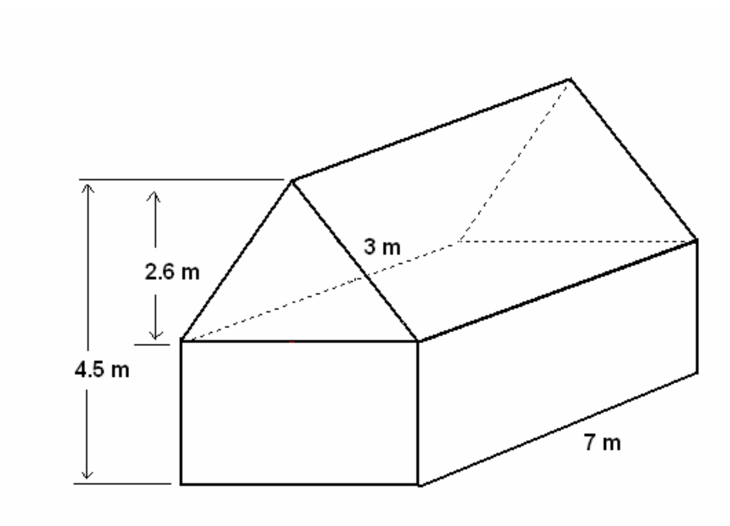
a)



b)



c)

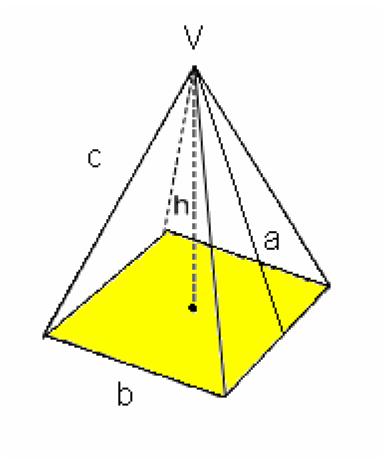


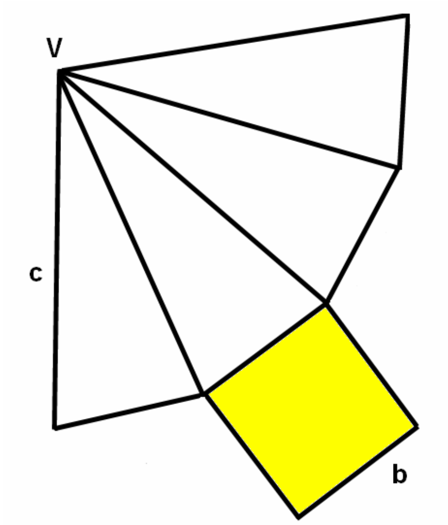
3 Pyramids

In a pyramid one of its faces is a polygon called the **base** and the other faces are triangles that join at a point that is the **apex.**

The height of a pyramid (h) is the distance from the base to the apex.

This is a s**quare pyramid**





3.1 Area of a pyramid

The area of a pyramid whose base is a regular polygon of **n** sides of length **b** and apothem **g,** and calling **a** to the height of the triangular faces (**slant height**) is:

$A = A_b + A_l$, The base is a regular polygon with area $A_b = \frac{n \cdot b \cdot g}{2}$ and there are n triangular faces with a total area of $A_l = \frac{n \cdot b \cdot a}{2}$

$$A = \frac{n \cdot b \cdot g}{2} + \frac{n \cdot b \cdot a}{2} = \frac{n \cdot b \cdot (g + a)}{2}$$

If it is a square based pyramid like the one in the picture, the area is $A = b^2 + 2 \cdot b \cdot a$

3.2 Volume of a pyramid

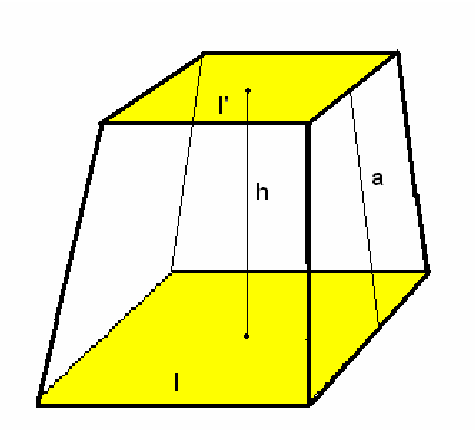
The volume of a pyramid is $V = \frac{\text{area of the base} \times \text{height}}{3}$

If it is a square pyramid like the one in the picture $V = \frac{1}{3} b^2 \cdot h$

3.3 Trunk of a pyramid

A trunk of a pyramid is the part of a pyramid which is between two parallel planes. The faces of the solid obtained by cutting it are called **bases** of the **trunk,** and the Distance between the two cutting planes is the **height** of the **trunk.** The side faces are trapeziums.

The area of a trunk of a pyramid with bases which are regular polygons of n sides, is



$A = A_{base1} + A_{base2} + n\frac{l + l'}{2}a$, being l and l' the sides of the base polygons and a the height of the trapeziums.

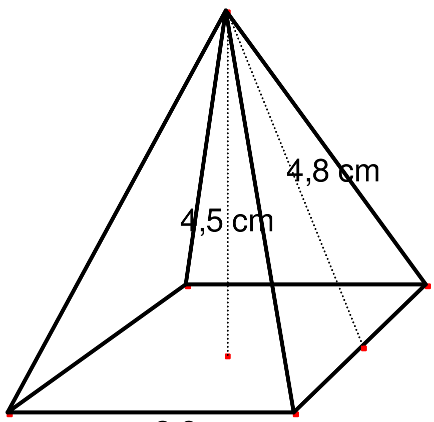
In the square based pyramid on the right the area of a trapezium is $A_t = \frac{(l + l') \cdot a}{2}$ and the area of the bases are $A_1 = l^2$ and $A_2 = (l')^2$

The area of the trunk is

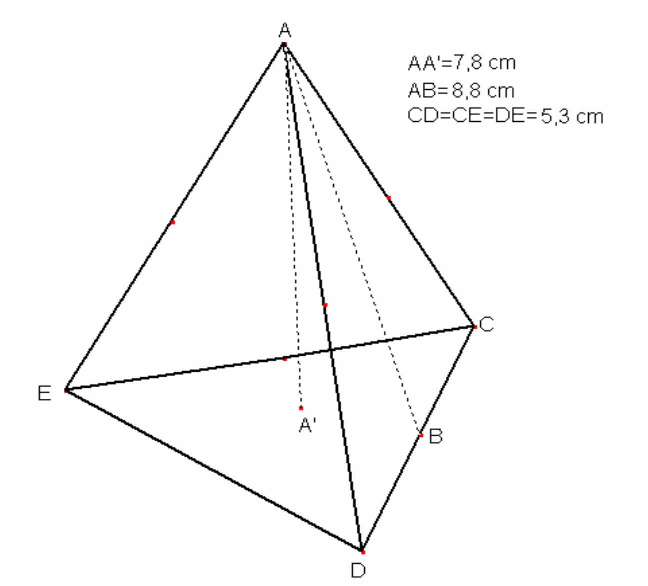
$A = 2(l + l') \cdot a + l^2 + (l')^2$

Exercise 5 Find the area and the volume of these pyramids:

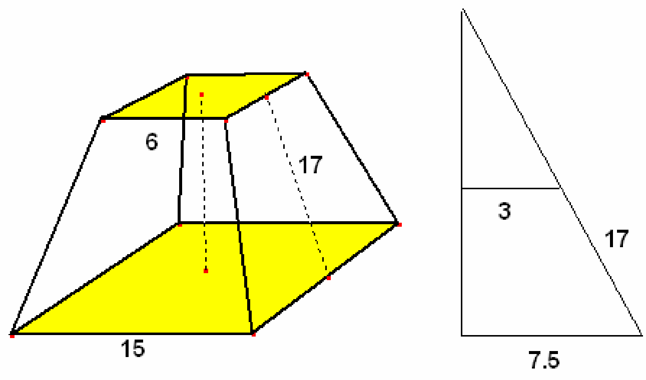
a)



b)

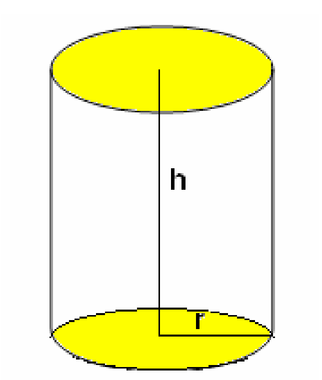


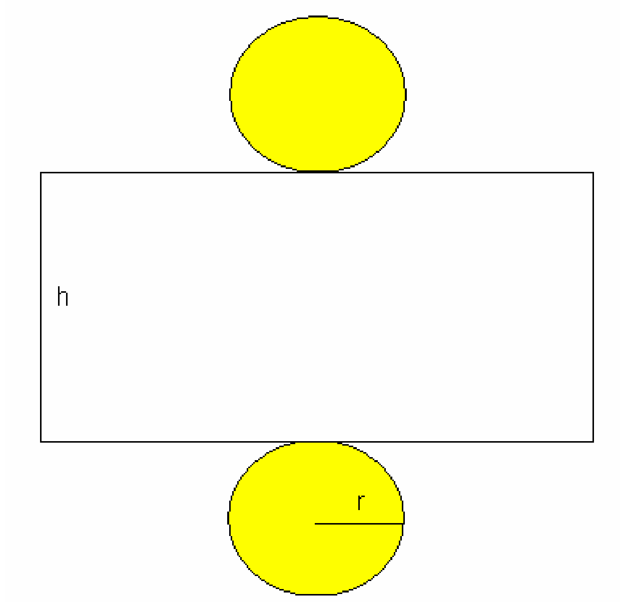
Exercise 6 Find the area and the volume of the trunk of pyramid shown below:



4 Cylinders

A **cylinder** is a curvilinear geometric solid formed by a **curved surface** and two circles as bases.





The curved surface unrolled is a rectangle that measures h by $2\pi r$; the radius of the cylinder is the radius of any of the two bases.

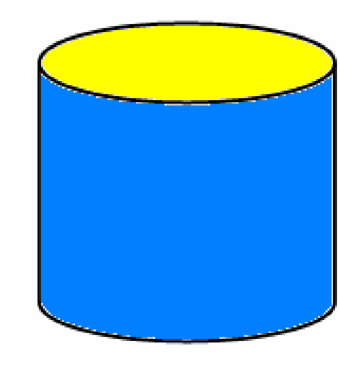
4.1 Area of a cylinder

The area of a cylinder is $A = 2\pi r h + 2\pi r^2$ and the volume is $V = \pi r^2 h$

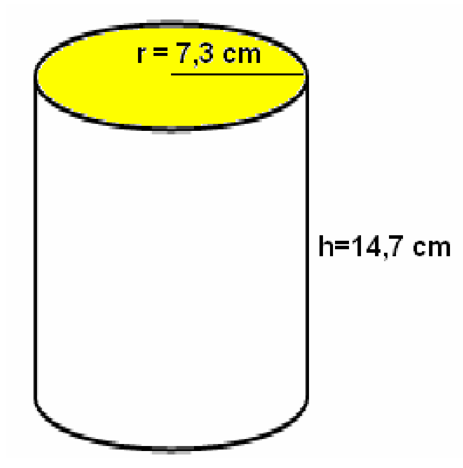
If the cylinder is an oblique cylinder, the formula for the volume is the same, but the perpendicular height is not equal to the height of the curved surface.

Exercise 7 Find the area and the volume of these cylinders

a) $r = 15$ cm $h = 12$ cm



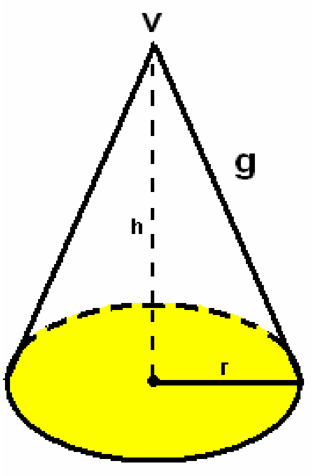
b)

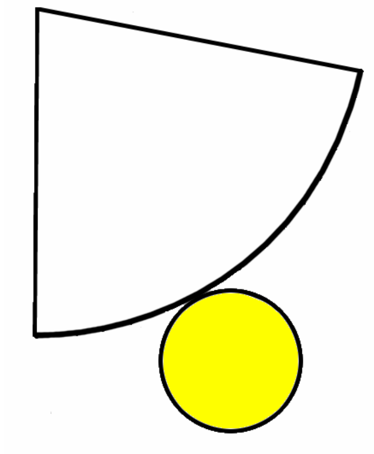


c) $r = 8$mm $h = 34$cm

5 Cones

A cone is a solid bounded by a curved surface that has a common point (vertex) and a circle as the base of the cone.





Vertex or apex is the top of the cone (**V**).

Slant height of the cone is the straight line that joins the vertex with the circle of the base (**g**).

5.1 Area and volume of a cone

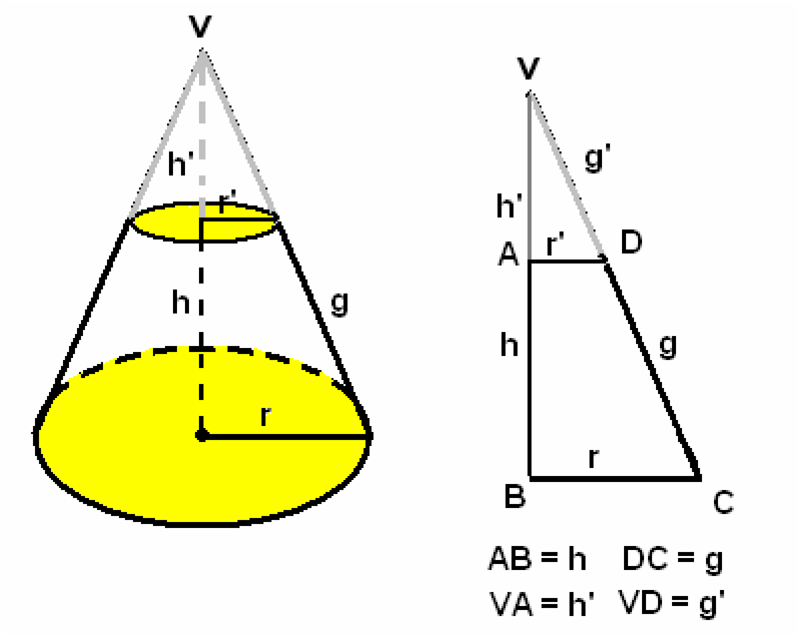
The area of the curved surface of the cone is the area of a sector and it is given by the formula $A = \pi r g$ and the area of the base is $A_b = \pi r^2$ so the total area of the cone is $A = \pi r g + \pi r^2$

The volume of the cone is $V = \frac{1}{3}\pi r^2 h$

5.2 Trunk of a cone

A trunk of a cone is the part of the cone which is between two parallel planes. The faces of the solid obtained by cutting it are called **trunk bases,** and the distance between the two cutting planes is the **height** of the **trunk.**

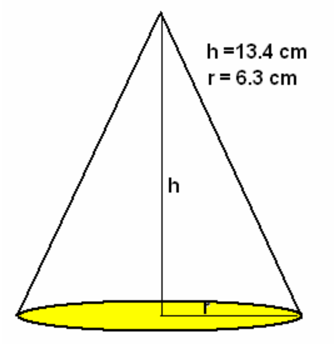
The area of any trunk of a cone is the area of the cone minus the area of the cone that has been removed



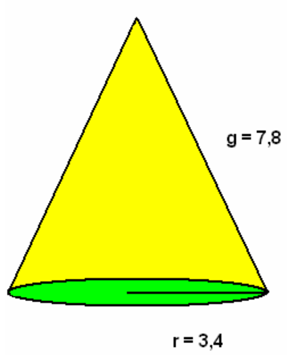
As can be seen in a vertical cut of the cone on the right the triangles VBC and VAD are similar and some of the measures can be calculated from the others.

Exercise 8 Find the area and the volume of these solids:

a)

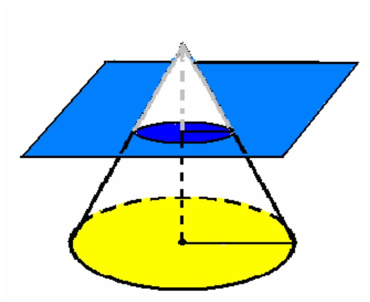


b)



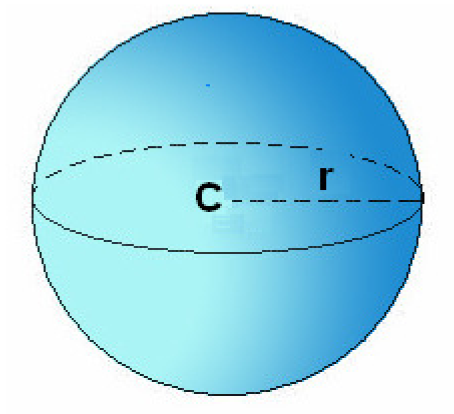
Exercise 9 The height of the cone of the picture is 38 cm and the radio r is 15 cm. It has been cut by a plane at 12 cm from the vertex. Calculate:

a) The area of the trunk of the cone
b) The volume of the trunk of the cone



6 Sphere

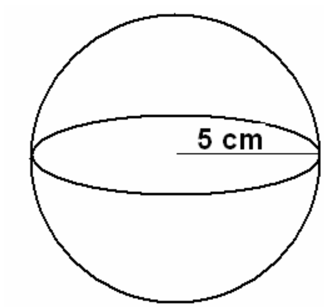
In a sphere all points are at the same distance **r** from the **centre** of the sphere **C**.
The distance from the centre to the surface of the sphere is called the radius of the sphere **r**



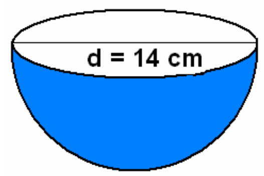
Area and volume of a sphere

The area of a sphere of radio **r** is $A = 4\pi r^2$ and the volume is $V = \frac{4}{3}\pi r^3$

Exercise 10 Find the area and the volume of the sphere



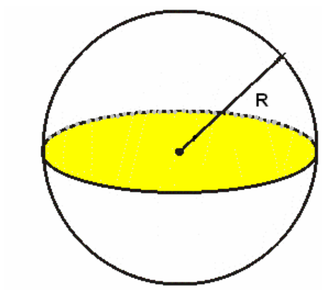
Exercise 11 Find the total area and the volume of this solid



Exercise 12 Find the area and the volume of an hexagonal prism where the base edge is 5 cm, the apothem 4 cm and the perpendicular height 13 cm

Exercise 13 Calculate the volume and the area of a squared-based right pyramid. The edge of the base is 13.2 cm and the perpendicular height is 17 cm long.

Exercise 14 The volume of this sphere is 32 cm^3. find its radius and its area.



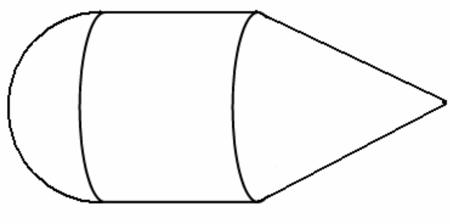
Exercise 15 The volume of a cone is 785.4 cm^3 and the perpendicular height is 10 cm. Find the area.

Exercise 16 One Egyptian pyramid has a base side of 112 m and a height of 130m. Work out which volume of stone would be needed to build this pyramid.

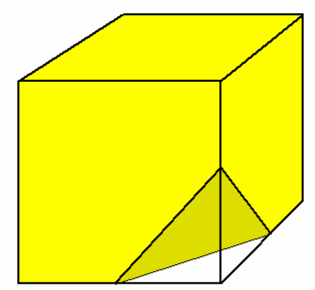
Exercise 17 A tomato tin is cylindrical. Each tin has a capacity of 1 litre and a base radius of 5 cm. Find the height and the area of the tin.

Exercise 18 The paddling pool of the picture is an hexagonal-based prism, the base is a regular polygon, the edge of the base is 1,5 m and the depth of the water is 60 cm. Calculate the volume of water that this pool contains. Calculate the weight of this water in Kg.

Exercise 19 A solid consists of a cylinder with a diameter of 15 cm and a height of 7 cm glued together to an hemisphere, on one side and to a cone, with a height of 10 cm, on the other. Find the total volume and area of the solid



Exercise 20 One corner of a 13 cm sided cube has been removed by cutting through the midpoints of the edges like in the picture. Find the volume and the area of the remaining piece.



Exercise 21 A rectangular sheet of paper 15 cm by 29 cm is rolled up to make a tube. Find the radius and the volume of the tube

a) **If the long sides are joined**
b) **If the short sides are joined**